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IND

BERRY, ARTHUR. On Elliptic Modular Equations for Transformations of Orders 29, 31, 37, ... 156

BOLZA, OSKAR. The Determination of the Conjugate Points for Discontinuous Solutions in the Calculus of Variations, ... 209

BUTTS, WILLIAM H. The Elliptic Cylinder Function of Class K, ... 129

CARMICHAEL, R. D. A Table of the Values of m Corresponding to Given Values of $\phi(m)$, ... 394

DICKSON, L. E. On the Canonical Forms and Automorphs of Ternary Cubic Forms, . 117

DICKSON, LEONARD EUGENE. Invariantive Reduction of Quadratic Forms in the $GF[2^n]$, ... 263

EIESLAND, JOHN. On Translation-Surfaces Connected with a Unicursal Quartic, . 179

EISENHART, LUTHER PFAHLER. Surfaces with the same Spherical Representation of their Lines of Curvature as Spherical Surfaces, ... 19

LASKER, E. A New Method in Geometry, ... 65

LEONARD, HEMAN BURR. On the Factoring of Composite Hypercomplex Number Systems, ... 43

LOVETT, EDGAR ODELL. On a Group of Transformations which Occurs in the Problem of Several Bodies, ... 307

MACMILLAN, WILLIAM DUNCAN. The Motion of a Particle Attracted Towards a Fixed Center by a Force Varying Inversely as the Fifth Power of the Distance, ... 282

MILLER, G. A. Groups Generated by n Operators Each of Which is the Product of the $n-1$ Remaining Ones, ... 93

RUSSELL, BERTRAND. Mathematical Logic as Based on the Theory of Types, . 222

SISAM, C. H. Concerning Systems of Conics Lying on Cubic Quartic and Quintic Surfaces, ... 99

SNYDER, VIRGIL. Plane Quintic Curves which Possess a Group of Linear Transformations, ... 1

SNYDER, VIRGIL. On Birational Transformations of Curves of High Genus, . . 10

SNYDER, VIRGIL. Normal Curves of Genus 6, and their Groups of Birational Transformations, ... 325

SNYDER, VIRGIL. On the Range of Birational Transformation of Curves of Genus Greater than the Canonical Form, ... 337

STEPHENS, R. P. On the Pentastroid, ... 381

VEBLEN, OSWALD, and YOUNG, JOHN WESLEY. A Set of Assumptions for Projective Geometry, ... 347

Plane Quintic Curves which Possess a Group of Linear Transformations.

By Virgil Snyder.

1. Plane curves which are invariant under linear transformations are important because they furnish examples of configurations of points of inflexion which disprove Grassmann's theorem.*

Thus far no general positive results have been obtained regarding the position and dependence of such points, and each new concrete case is of interest. The purpose of this paper is to discuss the plane quintic curves which are left invariant by linear transformations, without considering the important cases that are self-dual.

2. Finite ternary linear groups may be divided into three classes:

a) Those leaving a triangle invariant;
b) The regular body groups;
c) Ternary groups not belonging to a) nor b).

Of c), the only forms which have been found are G_{216} (Jordan), G_{168} (Klein), and G_{360} (Valentiner). The invariant form of G_{216} is the syzygetic pencil of c_3, that of the simple G_{168} is the c_4 defined by $x^3y + y^3z + z^3x = 0$, and its covariants of orders 6, 12, 21; that of the simple group G_{360} is a particular c_6 and its covariants of orders 12, 30, 45.†

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* "Zur Theorie der Wendepunkte, besonders der Curven vierter Ordnung," by Justin Grassmann, Dissertation, Berlin (1875). The theorem in question (p. 45) is: *If a c_{n-3} be passed through $\frac{1}{2}(n-2)(n+1)$ points of inflexion of a non-singular c_n, the remaining $\frac{1}{2}(n-1)(n-2)$ points of intersection will also be points of inflexion of c_n.* For $n = 4$, compare the remark by Klein, *Math. Ann.* Vol. 10, p. 397, (1876).

† For the literature concerning these groups, see Encyclopädie, IB 2, vol. 1_1, pp. 339–340, and IB 3f, vol. 1_1, pp. 528–529. To this list three recent papers by Ciani may be added: "Un teorema sopra la quartica di Klein," *Rend. Ist. Lombardi* (2), vol. 33 (1900) pp. 565-566. "I gruppi finite di collineazioni [illegible]," [illegible], pp. 1170-1175. "Contributo alla teoria del gruppo di 168 collineazioni piani," *Annali* (3), vol. 5, (1901) pp. 33-55.

Of the regular body groups, the dihedral group and the invariant G_4 of the tetrahedral group need not be considered, since they are contained in a). From the invariant forms of the remaining transformations it is seen that a c_5 cannot be transformed into itself, hence we need only consider groups of the form a). These may be either cyclical perspectivities, $x' = ax$, $y' = y$, $z' = z$, $a^k = 1$ or projections of rotations $x' = ax$, $y' = a^{-1}y$, $z' = z$, or combinations of the two. The order of the first cannot exceed 5, that of the second cannot exceed 10, but the third may have a higher order. In case of the cyclical perspective of order 5 the center cannot lie on the curve. The axis cuts the curve in points at each of which the tangent has five-point contact and passes through the center. No other tangents can be drawn from the center. If $k = 4$, the center is a simple point on c_5, at which the tangent has five-point contact. The axis cuts c_5 in five points at which the tangents have four-point contact and pass through the center. If $k = 3$, the center is a double point and each tangent has five-point contact. The tangents at the five points of inflexion on the axis pass through the center. Finally, if $k = 2$, the center is either a point of inflexion or a triple point, each branch having an inflexion at the point. In the former case, five tangents have their point of contact on the axis, and six bitangents pass through the center. In the latter case, only five tangents can be drawn from the center, and these have their points of contact on the axis.

It was shown by Maschke* that every curve which is invariant under a) is an integral function of xyz and $x^\alpha y^\beta + y^\alpha z^\beta + z^\alpha x^\beta = 0$, α, β being zero or positive integers. The invariant curves of the cyclic perspectivities will first be considered.

3. It can be easily shown that a c_5 cannot remain invariant under two harmonic homologies having a common axis. For, let $z = 0$ be the common axis, (0, 0, 1), (0, 1, 1) be the two centers.

The first homology is $\begin{pmatrix} x & y & z \\ x & y & -z \end{pmatrix}$, and the second is $\begin{pmatrix} x & y & z \\ x & y-2z & -z \end{pmatrix}$. If c_5 is invariant under the first, its equation can contain no odd powers of z,

$$z^4 f_1(x, y) + z^2 f_3(x, y) + f_5(x, y) = 0.$$

On making the second transformation and equating the coefficients of the new terms which appear to zero the equation is seen to be factorable.

---

[*] "On ternary substitution groups of finite order which leave a triangle unchanged," JOURNAL, vol. 17 (1895), pp. 168-184.

Moreover, two concentric harmonic homologies cannot leave a c_5 invariant from the remark concerning the configuration of the tangents and bitangents from the center which we saw must lie on the curve. If A, A' be the centers and a, a' the axes of two harmonic homologies which leave c_5 invariant, either A lies on a', A' on a, or a third homology exists whose center is on AA'. In the first case, by taking a, a', AA' for axes, the transformations may be expressed by

$$\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}, \quad \begin{pmatrix} x & y & z \\ x & -y & z \end{pmatrix}.$$

This requires that x, y enter the equation in even powers only, which is impossible. Hence if c_5 admits two such homologies it also admits a third; if there be a third, such that its center is not on the line joining the first two, we must have at least nine, forming a sort of Hessian configuration. Each center is a point of inflexion on the curve. The line joining three centers cannot be a tangent to the curve, nor pass through a double point.

4. We now introduce the following notation:

$$U_1 \equiv \begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}, \quad T_1 \equiv \begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}, \quad U \equiv \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix},$$

$$S_1 \equiv \begin{pmatrix} x & y & z \\ \omega x & y & z \end{pmatrix}, \quad \omega^3 = 1; \qquad I_1 \equiv \begin{pmatrix} x & y & z \\ ix & y & z \end{pmatrix},$$

$$V_1 \equiv \begin{pmatrix} x & y & z \\ \theta x & y & z \end{pmatrix}, \quad \theta^5 = 1, \qquad U = U_1 U_2,$$

with corresponding symbols for similar changes in y and z. Let P_k represent a a k-fold point on the quintic curve c_5, $f_k(x, y)$ be a binary form in x, y of degree k, and $\phi_k(x, y)$ a symmetric form, so that $\phi_k(x, y) = \phi_k(y, x)$. Let G_m be a group of order m, in which $\begin{pmatrix} x & y & z \\ ax & ay & az \end{pmatrix}$ is regarded as identity.

5. The c_5 having P_4 at $(1, 0, 0)$ and symmetric in y, z

$$x\phi_4(y, z) + \phi_5(y, z) = 0$$

is invariant under U_1. P_4 lies on the axis $y - z = 0$. The center $(0, 1, -1)$ is a point of inflexion from which can be drawn two bitangents; the simple tangent from the center has its point of contact at the residual point on the axis. In particular, three forms may be considered that are non-symmetric:

$$\begin{aligned} axy^4 + by^5 + cz^5 &= 0, \\ axyz^3 + by^5 + cz^5 &= 0, \qquad (1) \\ axy^4 + bz^5 &= 0. \end{aligned}$$

The first two have a cyclic G_5, generated by V_3, and the third a G_{20}, generated by I_2 and V_3. All the point and line singularities of this last curve are concentrated in the invariant points $(0, 1, 0)$, $(1, 0, 0)$.

The curve

$$axy^2 z^2 + b(y^5 + z^5) = 0 \tag{2}$$

has a G_{10}, generated by $V_1 V_2$ and U_1. It has five points of inflexion, all on $x = 0$, each one being the center of a harmonic homology. It has five bitangents, one through each point of inflexion. The curve

$$x(x^4 + y^4) + z(x^4 - y^4) = 0$$

is invariant under I_2. The tangent at $(0, 1, 0)$ has five-point contact. The axis $y = 0$ cuts c_5 in $(1, 0, -1)$ and at P_4. In the former the tangent has four-point contact. The four remaining inflexions lie on a line passing through the center, upon which they form a harmonic range.

6. When c_5 has a P_3 and is invariant under T_1 its equation becomes

$$x^2 f_3(y, z) + f_5(y, z) = 0.$$

In particular, the curve

$$Az^2(x^3 + ay^3) + Bx^2(x^3 + by^3) = 0 \tag{3}$$

is invariant under T_3 and S_2. The center $(0, 0, 1)$ of T_3 is a P_3 and $(0, 1, 0)$, the center of S_2, is a P_2. Each tangent at both multiple points has five-point contact. The line $y = 0$ cuts c_5 in two points of inflexion; the remaining 12 are arranged in two hexads. The tangents at P_2 each count for one double tangent; the remaining 30 are grouped in five hexads.

The curve

$$ax^2 y^3 + bz^5 = 0 \tag{4}$$

is invariant under T_1, S_2, V_3, hence has a G_{30}. The two binomial curves are self-dual. They also have a continuous group.

7. The curve $x^3 f_2(y, z) + f_5(y, z) = 0$ has S_1, and

$$x^3 \phi_2(y, z) + \phi_5(y, z) = 0 \tag{5}$$

allows S_1 and U_1. The lines $f_5(y, z) = 0$ are all inflexional tangents passing through $P_2 \equiv (1, 0, 0)$, the points of inflexion lying on $x = 0$. The remaining 30 inflexions are arranged in groups of six, a line joining P_2 to any one will contain two others, and the line joining any one to (0, 1, — 1) will contain one other.

The curve

$$ax^3y^2+by^5+cz^5=0 \tag{6}$$

has a G_{15}, generated by S_1 and V_2. It has a cusp and adjacent double point at (1, 0, 0), the tangent $y=0$ having five-point contact. The side $z=0$ cuts c_5 in three points at which the tangents have five-point contact and pass through (0, 0, 1). The side $x=0$ cuts the curve in five ordinary inflexions, the tangents passing through the singular point. The latter absorbs two inflexions and two double tangents. The remaining 15 inflexions form a set, of which one is real. The 45 remaining double tangents are arranged in three such sets. If y^5 be replaced by y^4x, the curve has a G_{10} of type (2).

The curve

$$ax^3yz+b(y^5+z^5)=0 \tag{7}$$

is invariant under U_1, S_2 and $V_1 . V_3^2$, generating a G_{30}. The curve has a P_2 at (1, 0, 0), each tangent having five-point contact. $x=0$ cuts the curve in five ordinary inflexions and the remaining 30 are arranged in two sets, conjugate under U_1. Only two are real. The 30 points are also situated on 10 lines passing through the node. Each point of inflexion on $x=0$ is the center of a harmonic homology; six bitangents pass through each.

8. The curve $x^4f_1(y, z)+f_5(y, z)=0$ has the center (1, 0, 0) for a triple inflexion; four double inflexions lie on the axis of I_1 and the remaining 32 are arranged in eight harmonic groups on lines passing through the center.

The curve

$$ax^4y+b(y^5+z^5)=0 \tag{8}$$

is invariant under V_3 and I_1, making a G_{20}. The five points of c_5 on $x=0$, have four-point contact tangents, passing through (1, 0, 0) at which the tangent has five-point contact. Each point on $z=0$ has a five-point contact tangent. The remaining 20 form one cycle, two of them being real. They are arranged on four lines through (0, 0, 1), and on five through (1, 0, 0). No proper bitangent passes through the center of I_1.

9. The curve

$$x^5+\phi_5(y, z)=0 \tag{9}$$

belongs to V_1 and U_1. Five points of five-point contact tangents lie on $x=0$, and the remaining points of inflexion are arranged on six lines, any three of which compose an improper c_3, whose complete intersection with c_5 consists of points of inflexion, as does also the intersection of c_5 and $x^3 = 0$.

10. The curve

$$z^3 y^2 = x(x^4 + y^4) \tag{10}$$

has the G_{12} generated by $S_3 \,.\, T_3 \,.\, I_2$. It also has a Cremona group of order 72. It has a tacnode cusp at (0, 0, 1).

11. Of curves of the type $x^\alpha y^\beta + y^\alpha z^\beta + y^\alpha x^\beta = 0$, $\alpha, \beta \neq 0$ there are but two types to be considered.

The curve

$$x^4 z + z^4 y + y^4 x = 0 \tag{11}$$

is invariant under

$$U \equiv \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix} \text{ and } K \equiv \begin{pmatrix} x & y & z \\ x & \varepsilon y & \varepsilon^4 z \end{pmatrix}, \qquad \varepsilon^{13} = 1.$$

At each vertex of the triangle of reference a side has four point contact. U is the product of

$$K_1 \equiv \begin{pmatrix} x & y & z \\ x & \varepsilon y & \varepsilon^{-1} z \end{pmatrix} \text{ and } K_2 \equiv \begin{pmatrix} x & y & z \\ x & y & \varepsilon^5 z \end{pmatrix}.$$

The first is the projection of a rotation about (1, 0, 0), $y = 0$, $z = 0$ being the isotropic lines, and K_2 is a similarity transformation about (0, 0, 1), through which a real point goes into an imaginary point and return to its original positions after 12 operations.* Since reality is not changed by K_1 it follows that there can be at most three real points of inflexion, apart from those at the vertices, but as U is a real transformation, there must be just three.

The other curve of this type is

$$x^3 z^2 + z^3 y^2 + y^3 x^2 = 0; \tag{12}$$

it has a cusp of the first kind at each vertex and is invariant under U and

$$R \equiv \begin{pmatrix} x & y & z \\ x & \beta^2 y & \beta^3 z \end{pmatrix}, \ \beta^7 = 1.$$

---

* The theorem that if a point P be successively transformed into $P_1, P_2, \ldots P_{n-1}, P$ by a cyclic collineation of order n, the points P_i all lie on a conic is not necessarily true when the collineation is not real. In both statements of the theorem, a real collineation is tacitly presupposed. See J. Lüroth, "Das Imaginäre in der Geometrie und das Rechnen mit Würfen," *Math. Annalen*, vol. 11, p. 84, and "Ueber cyclisch-projectivische Punktgruppen . . . ," ibid, vol. 13, p. 304. A. Ameseder, "Theorie der cyclischen Projectivitäten," *Wiener Sitzungsberichte*, vol. 98, IIa, p. 290.

It has 21 points of inflexion of which three are real. Neither of these curves is invariant under any other collineation.

12. The most general c_5 which allows the three harmonic homologies U_1, U_2, U_3 is.

$$A\sigma_1^5 + B\sigma_1^3\sigma_2 + C\sigma_1^2\sigma_3 + D\sigma_1\sigma_2^2 + E\sigma_2\sigma_3 = 0, \tag{13}$$

wherein σ_i is an elementary symmetric function of x, y, z of weight i. The centers of these homologies lie on $\sigma_1 = 0$ and their axes intersect in (1, 1, 1), pole of $\sigma_1 = 0$ as to the triangle of reference. σ_1 cuts c_5 in three points of inflexion, and two other points defined by $x^2 + xy + y^2 = 0$, which are the same for all curves of the system. The tangents at the points of inflexion will all pass through the pole if $3D + E = 0$. In this case all curves of the net have 13 points in common. The tangents at the simple intersection of σ_1, σ_5 are $x + \omega y + \omega^2 z = 0$. They intersect at (1, 1, 1) for every curve of the net.

Each conic of the pencil

$$(x + \omega y + \omega^2 z)\,(x + \omega^2 y + \omega z) - k\,(x + y + z)^2 = 0$$

is invariant in the group generated by U_1, U_2, U_3, hence if it pass through a point of inflexion it will also pass through five others. *The 42 points of inflexion are arranged by sizes on seven conics belonging to a double contact pencil. The other three are on the line joining the points of contact.* These points on each c_2 are in three-fold involution, having $\sigma_1 = 0$ for Pascal line. Similarly, a conic of the pencil touching a double tangent will touch five others; they form a system in triple involution, having (1, 1, 1) for Brianchon point. The 24 remaining common tangents to c_2 and c_5 form four perspective quadrilaterals.

The six common tangents of a set may be expressed by t, tU_1, tU_2, tU_3, tU_1U_2, tU_1U_3.

Of the 15 intersections of these lines, three lie on each axis, and the remaining six lie on a conic of the pencil, defining a three-fold involution.

13. An interesting particular case is furnished by the curve

$$x^5 + y^5 + z^5 = 0 \tag{14}$$

which is also invariant under V_1, V_2, V_3 making with U_1, U_2, U_3 a group of order 150.*

---

* For a discussion of the G_{96} leaving a c_4 of similar equation invariant, see Dyck "Notiz über eine reguläre Riemannsche Fläche von Geschlecht drei und die zugehörige Normalcurve vierter Ordnung.' *Math. Ann.* vol. 17, (1887), pp. 510–516.

The 45 inflexions are arranged by fives on the sides of the triangle of reference, each one counting for three; the five-point-contact tangents pass through the opposite vertices. Here there are but two conics of the pencil

$$(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) - k(x + y + z)^2 = 0$$

which contain points of inflexion, since these two conics and the line $\sigma_1 = 0$ contain all 15 points.

The line joining any two points of inflexion will always pass through a third. There are 25 such lines; their equations are of the form

$$x + \theta^l y + \theta^k z = 0 \qquad l, k = 1, \ldots 5.$$

If we denote the point on $x = 0$ in which $y + \theta^i z = 0$ cuts it by x_i and similarly for y, z and finally replace x_i, y_k, z_l by (i, k, l) we may say: Three points x_i, y_k, z_l lie on a line when $i + k + l \equiv 0 \pmod{5}$. The equation of the line may then be written $x + \theta^l y + \theta^k z = 0$, and in five other equivalent forms. The lines (4, 4, 2), (1, 3, 1); (1, 2, 2), (4, 3, 3); (4, 2, 4), (1, 1, 3); (3, 3, 4), (2, 1, 2); (2, 4, 4), (3, 1, 1); (3, 4, 3), (2, 2, 1) intersect on $x - y$. Of these 12 lines, each intersects two others in points not at inflexions nor on an axis of homology. By U_1, the sum of the second and third symbols in any line remains constant, and symmetrically for the others. By transforming U_1, U_2, U_3 through V_1, etc., we see that G_{150} *contains* 15 *harmonic homologies*. Each of the 25 lines is invariant in three homologies whose centers are the three points of inflexion lying upon it. The three axes of homology pass through the pole of the line of centers as to the triangle of reference, from which two tangents can be drawn to c_5, meeting it at the residual intersection of (i, k, l). The pencil of conics in each case remains invariant, hence: *The fifteen points of inflexion are arranged by sixes on* 50 *conics, by threes on* 25 *lines and by fives on three lines.*

Through the nine points of inflexion lying on any three lines a pencil of c_3 can be passed, but no curve of any pencil can contain a tenth point of inflexion without becoming the sides of the triangle of reference. Similarly for the pencil formed by any line and either of inflexional conics, as

$$xyz + \lambda(x + y + z)c_2 = 0.$$

Similar configurations exist for the bitangents. Three are absorbed in each inflexional tangent, and five proper ones pass through each point of inflexion.

14. Of the preceding types, those which are invariant under more than one cyclic perspective are here arranged for reference.

$$axy^4 + bz^5 = 0. \qquad G_{20}. \tag{1}$$

$$axy^2z^2 + b(x^5 + z^5) = 0. \qquad G_{10}. \tag{2}$$

$$az^2(x^3 + ay^3) + bx^2(x^3 + by^3) = 0. \qquad G_6. \tag{3}$$

$$ax^2y^3 + bz^5 = 0. \qquad G_{30}. \tag{4}$$

$$x^3\phi_2(y, z) + \phi_5(y, z) = 0. \qquad G_6. \tag{5}$$

$$ax^3y^2 + by^5 + cz^5 = 0. \qquad G_{15}. \tag{6}$$

$$ax^3yz + b(y^5 + z^5) = 0. \qquad G_{30}. \tag{7}$$

$$ax^4y + b(y^5 + z^5) = 0. \qquad G_{20}. \tag{8}$$

$$x^5 + \phi_5(y, z) = 0. \qquad G_{10}. \tag{9}$$

$$z^3y^2 - x(x^4 + y^4) = 0. \qquad G_{12}. \tag{10}$$

$$x^4z + z^4y + y^4x = 0. \qquad G_{39}. \tag{11}$$

$$x^3z^2 + z^3y^2 + y^3x^2 = 0. \qquad G_{21}. \tag{12}$$

$$A\sigma_1^5 + B\sigma_1^3\sigma_2 + C\sigma_1^2\sigma_3 + D\sigma_1\sigma_2^2 + E\sigma_2\sigma_3 = 0. \qquad G_6. \tag{13}$$

$$x^5 + y^5 + z^5 = 0. \qquad G_{150}. \tag{14}$$

Among these types, the only ones that have more than one harmonic homology are (2), (7), (13), (14), the numbers being respectively 5, 5, 3, 15.

CORNELL UNIVERSITY, *August* 7, 1906.

On Birational Transformations of Curves of High Genus.

By Virgil Snyder.

The purpose of this paper is to show that nonsingular curves and others of genus exceeding a given number cannot be transformed into other curves of the same order by birational transformations other than collineations. The method employed will be two-fold; for the nonsingular curves we consider sections of a certain ruled surface then establish it for higher cases by induction. For the other cases, the "n-gonal" series of Bertini and certain inequalities will be employed in connection with linear transformations of hyperspace.

1. Given two nonsingular plane curves of order four, c_4, c_4' in $(1, 1)$ point correspondence such that A, A' and B, B' are two pairs of corresponding points. Find c_4'', projective with c_4', such that $A'' \equiv A$, $B'' \equiv B$, but otherwise unrestricted. Turn the plane of c_4'' about AB through any angle. The lines joining pairs of corresponding points P, P'' will generate a ruled surface of order 6 and genus 3. But by Wiman's formula*

$$p \leqq \tfrac{1}{6}(n-2)(n-3)$$

wherein p is the genus, n the order of a ruled surface not contained in a linear congruence. Hence the sextic must belong to a linear congruence and the only quartic curves upon it lie in the planes of the pencil whose axis is a double generator. From the method in which the surface was generated, the points C, D in which AB cuts c_4 and C'', D'' in which it cuts c_4'' must be corresponding points no matter what points were chosen for A, B. Hence c_4, c_4'' are projective.†

For nonsingular c_6' and c_6 exactly the same reasoning can be employed; for $n \geqq 7$, however, no new results are obtained, since the resulting ruled surfaces are of too high an order to preclude the possibility of the required genus.

---

* A. Wiman, "Klassification af regelytorna af sjette graden." Dissertation, Lund, 1892. See also *Acta Mathematica*, vol. 19 (1893).

† This proof for c_4 was given by me in the Journal, vol. 25 (1904), p. 187, but only too brief an outline of its extension to higher orders was there given.

2. From the Brill-Noether theorem we know that if any c_n is birationally transformed into c_N, the adjoint curves ϕ_{n-3} go over into ϕ_{N-3}, and conversely every such transformation which transforms the entire system ϕ_{n-3} into a system ϕ_{N-3} will transform c_n into some c_N. From what we have just seen, no transformation except collineations can transform the $\infty^2 . c_1$, the $\infty^5 c_2$ or the $\infty^9 c_3$ into themselves.

If a c_7 of $p = 15$ can be transformed into itself or any other c_7, the ∞^{14} system of ϕ_4 must also remain invariant; but this system may be defined by fourteen nonsingular curves of the system, which must be linearly transformed.

$$\phi_4 = \Sigma \phi_i . a_i .$$

From § 1 this is only possible by collineations. In the same manner for c_8, since we can define the adjoint system by nonsingular ϕ_5, and for c_9, ϕ_6. Since the theorem is true for c_κ, $c_{\kappa+1}$, $c_{\kappa+2}$ by repeating this process, it is true for all nonsingular curves, hence:

When a (1, 1) *correspondence can be established between the points of any two nonsingular plane curves, they are projectively equivalent.*[*]

3. In case of c_5 of $p = 5$, ϕ_{n-3} are conics passing through the node. If this point be (0, 0, 1), the equation of the system may be written

$$a x^2 + 2 h x y + b y^2 + 2 g x z + 2 f y z = 0.$$

It must go into itself by all the birational transformations of c_5. If we put

$$\rho x_1 = x^2,\ \rho x_2 = x y,\ \rho x_3 = y^2,\ \rho x_4 = x z,\ \rho x_5 = y z$$

and regard x_i as homogeneous point coordinates in a linear space of four dimensions R_4, the image of c_5 is a c_8, whose intersections with R_3 are the eight points common to

$$x_2^2 = x_1 x_3,\ x_3 x_4 = x_2 x_5,\ x_1 x_5 = x_2 x_4 . \dagger$$

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[*] This theorem presupposes $n > 3$. Two cubic curves in (1, 1) correspondence are projectively equivalent as a whole, though any nonsingular c_3 can be birationally transformed into itself by an infinite number of nonlinear transformations. For $n = 4$ the result immediately follows from consideration of the adjoint curves, which are straight lines. The theorem is stated without proof for $n = 5$ and $n = 6$ by Wiman, "Ueber die algebraischen Kurven von den Geschlechtern $p = 4$, 5, 6, welche eindeutige Transformationen in sich besitzen," *Stockholm Akademien Handlingar*, Bihang, vol. 21, no. 3, pp. 1–43 (1895). The general theorem is similarly stated by me in the JOURNAL, *l. c.* and also by C. Küpper, "Ueber das Vorkommen von linearen Schaaren g_n^2 auf Kurven n^{ter} Ordnung . . .," *Prager Sitzungsberichte*, 1892, p. 264.

[†] While this depiction has been extensively employed by various writers, it has apparently not been used for the present purpose. The number of linearly independent quadratic relations among the ϕ curves was found by Weber, "Ueber gewisse in der Theorie der Abelschen Funktionen auftretende Ausnahmefälle," *Math. Ann.* vol. 13 (1877). Other particular cases were discussed by Kraus in *Math. Ann.* vol. 16.

But not every curve of genus 5 can be reduced to a nodal quintic, hence we should expect certain relations in consequence of the reduced number of moduli.*

In this case we have

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_4}{x_5},$$

hence the quadrics have a ruled hypersurface in common, having $x_1 = x_2 = x_3$ for directrix, and generators of the form $x_1 = \lambda x_2$, $x_2 = \lambda x_3$, $x_4 = \lambda x_5$. The directrix is the image of the node, and the generators are the images of the straight lines passing through it. Every linear transformation in R_4 which leaves the system of quadrics invariant must also transform this ruled hypersurface into itself; incidentally, therefore the directrix must remain fixed and the generators can only be permuted among themselves. The node (0, 0, 1) and the pencil $x = \lambda y$ of our plane c_5 therefore go into themselves. Among the ∞^4 adjoint conics are ∞^3 degraded ones, consisting of a line through the node and any other line, which must go into themselves; that is: *A nodal plane quintic curve can be transformed into itself or any other nodal quintic only by collineations.*

4. Exactly the same method will apply to curves of any order greater than 4 and having a single node. A number of the quadrics in R_n will have an invariant configuration in common which is the image of the node and the pencil of straight lines through it. As the ∞^{p-1} system of ϕ_{n-3} can be transformed into itself by collineations only (by § 1), we may say: *When two curves* ($p > 2$) *having a single node are in* (1, 1) *point correspondence, they are projectively equivalent.*

5. Moreover, we can draw the same conclusions from curves of order n and having a single multiple point P_i of order $2 \leqq i \leqq n-3$. In case $i = n-3$, $p = 2n-5$ and the $(2n-7)(n-4)$ quadrics have a ruled hypersurface in common, whose $(n-3)$ fold directrix is the image of P_i and the generators are the images of the lines through it. The adjoint ϕ_{n-3} have an $(n-2)$ fold point, hence the equation is of the form

$$u_{n-3}(x, y)\, z + u_{n-2}(x, y) = 0.$$

The only transformation that will transform this system into itself is a collineation.†

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* The normal curves for general moduli are given for every value of p in Clebsch-Lindemann's Geometrie, vol. 1, p. 709.

† See Jung, "Ricerche sui sistemi lineari di curve algebriche di genere qualunque," *Ann. di Mat.* vol. 15 (1887), and vol. 16 (1888). This theorem has been employed by Kantor and by Wiman in their enumeration of finite groups of Cremona transformations.

If two plane curves of order n have a single multiple point of order $i \leqq n - 3$, *they can have no* (1, 1) *transformation into each other except by collineation.*

Such curves have a linear series g^1_κ, and when $i = n - 3$, they are particular cases of Bertini's trigonal curves.*

Moreover, this result can also be derived from Zeuthen's formula

$$c - c' = 2e'(p-1) - 2e(p'-1).$$

Here $e = 3$, $e' = 1$, $c' = 0$, $p' = 0$, hence $c = 2p + 4 = 4n - 6$ for the number of curves cutting g^1_3 which touch the given curve; but this is exactly the number of tangents to c_n from P_{n-3}. In every case the three points of each group of g^1_3 are collinear, and the lines are always concurrent if $p = 2m - 5$. If $p > 4$, evidently c_n cannot have two g^1_3, no matter what its configuration of double points may be, for if $\phi + \mu\phi' = 0$, $\psi + \lambda\psi' = 0$ be two systems of adjoints of order $n - 4$, between μ, λ would exist a (3, 3) relation, but such a correspondence has a maximum genus 4. No curve of order greater than 4 can have general moduli and possess a g^1_3.

6. A c_6 having two nodes has a system of $\infty^7\, c_3$ with the nodes for common basis points for adjoint curves. Among these curves are ∞^4 which factor into a line through each node and one other line. The pencils through the nodes must either remain invariant or simply interchange, hence the third line must go into a line.

A binodal sextic curve cannot be transformed into another binodal sextic except by collineation.

By means of §§ 1, 2 we can conclude in general: *When between the points of two binodal curves* ($p > 4$) *a* (1, 1) *correspondence exists, the curves are projectively equivalent.*

Now suppose c_n has a κ-fold point, $\kappa \leqq n = 4$, and a double point, but no other singularities. The $g'_{n-\kappa}$ through P_κ and the g'_2 through P_2 must both remain invariant, and the residual ϕ_{n-5} having a $(\kappa - 2)$-fold point must go into a similar curve. From the last preceding case it is only possible by collineations when $n = 6$. By § 2, this is also true for $n = 7$ and $n = 8$. From three consecutive sets we may build up any case, hence: *When a* c_n *has a* κ*-fold point* ($\kappa \leqq n - 4$) *and a double point it can be transformed into another* c_n *only by collineations.*

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* Bertini, "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico," *Ann. di Mat.* vol. 22 (1894), pp. 1–40. The result of this § could be obtained from *g*) p. 31.

7. Two fundamental questions arise from the preceding theorems: *a*) What is the largest number of double points a curve may have and not have a group of birational transformations except collineations? *b*) What is the lowest order of a curve to which a nonsingular curve may be transformed by birational but not by linear transformations? We shall answer these two questions in the order in which they are given.

8. On any curve the straight lines of the plane cut out a series g^2_n. Whenever it is possible to define another g^2_n, whose groups are not on lines, the curve can be transformed into another curve of the same order.

If δ be the largest number of double points such a curve can have, δ is less than the minimum for a space curve of the order n, since the latter has g^3_n formed by the ∞^3 planes of space. By projecting upon a plane, only those groups will be collinear points whose planes pass through the center of projection. Hence every such curve will have proper g^2_n. First then

$$p > \tfrac{1}{2}(n-1)(n-2) - \left[\left(\frac{n-1}{2}\right)^2\right].$$

For such values of p, no g^3_n exist, hence g^2_n are complete series. We are only concerned with $n > 6$, hence g^2_n is a special series, and can therefore always be cut from c_n by a proper or composite ϕ_{n-3}. If one such G_n be given, and a ϕ_{n-3} be passed through it, the residual points of intersection of ϕ_{n-3}, c_n form the basis points of ∞^2 such ϕ_{n-3}; the variable intersections then constitute the g^2_n to which the given G_n belongs (Riemann-Roch theorem). It was shown by Kantor* that if these ∞^2 ϕ_{n-3} are all composite, they must consist of one fixed curve and a variable curve of order x, the latter constituting an irreducible net.

Let d double points of c_n be among the basis points of the net. Of the nx intersections of c_n, c_x, n are to be variable and d lie in the double points. The number of fixed basis points of the net is therefore $n(x-1)-d$. It has been shown by Küpper† that the maximum number of fixed basis points of ∞^2 c_x is $x^2-(x-1)$, hence

$$d \geqq (x-1)(n-x)-1$$

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*S. Kantor, "Neue Theorie der eindeutigen periodischen Transformationen in der Ebene," *Acta Math.* vol. 19 (1895), pp. 115–193.

† C. Küpper, in the *Prager Abhandlungen*, series 7, vol. 3 (1889).

The c_x which determine g_n^2 on c_n must pass through at least $(x-1)(n-x)-1$ double points on c_n.

9. It is therefore necessary to take for x the value which will make d a minimum, provided such a curve is possible. A few illustrations will be given.

If $n=6$, $\delta=3$. Evidently any trinodal c_6 can be transformed into a c_6 by quadric inversion, using the nodes for fundamental points. No such conic is possible, however, if the three nodes are collinear, or coincident (§ 5).

If $n=7$, $\delta=7$. Evidently ϕ_{n-4} are here the ∞^2 c_3 through the seven nodes. If the double points are coincident, $\delta<7$. c_7 with $2P_3$ or P_3+2P_2 can be transformed into similar c_7 by quadratic inversion. If a P_4 be present, no g_x^2 exists unless a double also exists, i. e. $\delta=7$.

If $n>7$, $x>\frac{n-1}{2}$, since c_x must contain at least $(x-1)(n-x)-1$ double points of c_n among its fixed basis points. When $n=8$, $x=4$, and $\delta=11$. These points can be assumed at will, but the resulting curve is a particular one of order 8 and genus 10.

If $n=10$, x may be 7, 6, 5. The corresponding values of δ are 17, 19, 19, but $\delta=17$ leads to a contradiction. Through $17P_2$ and a G_{10} we can pass a c_6, but when $x=6$, $\delta=19$. If c_{10} has P_4+P_5 $(\delta=16)$ or P_4+2P_3 $(\delta=12)$ the transformation is possible, but not for P_7, $\delta=21$ (§ 5). To construct a c_{10} having 19 double points and a g_{10}^2, pass two c_5 through any four fixed points on c_1. These c_5 will intersect in 21 further points which are basis points of a net of c_5. Let 19 of these be chosen as double points on c_{10}; through them and five points on the same c_1 pass two c_6. We now have two pencils $c_5+\lambda c_5'=0$, $c_6+\mu c_6'=0$. Let λ, μ be so chosen that the c_5, c_6 passing through a given point will be corresponding. If the point be chosen on c_1, the correspondence will be (1, 1) because c_5 has 4 points on c_1 fixed, and c_6 has 5. The remaining locus will be c_{10} having $19P_2$. The net of c_5 or the net of c_6 will each cut from it a g_{10}^2.

10. Since $\frac{n}{2}\leqq x\leqq n-3$, and δ is small for larger values of x, it easily follows that *the largest number of distinct double points which a curve of order n may have without being birationally transformable into another curve of the same order is two less than the minimum number of double points which a space curve of the same order can have.* $(n>7)$.

11. For large values of n, only very particular curves c_n can be so transformed. To obtain a more precise limit, express the condition that a G_n cannot lie on c_{x-1}.

$$(x-1)(n-x)-1+x \leqq \tfrac{1}{2}(x-1)(x+2).\text{[*]}$$

Hence $x < \dfrac{2(n-1)}{3}$, from which we can say:

A c_n containing a g_n^2 not lying on the ∞^2 lines of the plane cannot have less than $(x-1)(n-x)-1$ double points, x being the largest integer less than $\dfrac{2(n-1)}{3}$.

Thus for $n=11$, $\delta=24$. It is an interesting exercise to construct this c_{11}. The double points cannot be assumed at will; they are among the 31 intersections of two sextics having 5 points on a straight line in common. A c_{11} with $2P_5$ ($\delta=20$) can be transformed into a similar c_{11} by inversion.

12. We now consider question (b). A general curve of genus p has $3p-3$ moduli which are undisturbed by birational transformation, but the most general c_n has not more than $\dfrac{n(n+3)}{2}-8$ or $p+3(n-3)$ such moduli. Any non-singular c_n can by quadratic inversion be transformed into a c_{2n-3}, having three $(n-2)$-fold points and no other singularities, and conversely.

Given two curves, c_x, c_y, $x \leqq y$. Their xy intersections are such that every c_{x+y-3} through all but one of them will contain that one also (Cayley). It is a minimum group on the curve. If $x+y=m+3$, then $x \leqq \dfrac{m+3}{2}$. The number of points in the group varies from $m+2$ to $\left(\dfrac{m+3}{2}\right)^2$ if m is odd, or to $\dfrac{(m+2)}{2}\cdot\dfrac{(m+4)}{2}$ if m is even. Our problem may now be stated thus: to find the smallest value of κ for which g_κ^2 exists on c_n, $\kappa > n$. It has been partially treated by Küpper[†] and his definitions will be here reproduced.

G_Q is called normal as to c_m if any c_m through $Q-1$ of the points does not have to pass through the other also, otherwise G_Q is abnormal as to c_m.

If G_Q is determined by $Q-q$ conditions, q is called the excess of G as to c_m.

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[*] It was shown by Küpper, *Prager Berichte* (1892), that if c_g contains a G_n of g_n^2 it imposes exactly $g+1$ conditions. This inequality is then proved, but a large number of errors have made the application there made of it of small value.

[†] C. Küpper, "Zur Theorie der algebraischen Curven," *Monatshefte für Math. und Physik*, vol. 6 (1895), pp. 127–156.

By the Riemann-Roch theorem, q is the number of degrees of freedom in the series of curves having the residual of G_Q for basis points. If G_Q is abnormal as to c_m, but every G_{Q-1} contained in it is normal, G_Q is called primitive. The following theorem can now be easily proved: *If it be impossible to pass c_i through G^q_Q, then the excess of the group as to c_{m-i} is $\frac{1}{2}(i+1)(i+2)$.*[*]

An abormal G_Q containing the smallest number of points is called a minimum group as to c_m. We can now prove the theorem:

Given a primitive G_{xy} and $x + y = m + 3$, $x \leqq y$; G_{xy} is a minimum group for c_m unless it lies on c_i $(i < x)$.

We first prove the following lemma: If n is the lowest order a curve can have which contains G_Q, and $n > \frac{m+3}{2}$ say $n = \frac{m+3+\delta}{2}$, then $Q > \left(\frac{m+3}{2}\right)^2$.

First, $2n = m + 3 + \delta$ or $n - 1 = m - (n - 2 - \delta)$.

On putting $n - 2 - \delta = i$, the excess of G_Q as to c_{n-1} is $\geqq \frac{(n-1-\delta)(n-\delta)}{2}$. If now c_{n-1} through G_Q is impossible

$$\frac{(n-1)(n+2)}{2} + \frac{(n-1-\delta)(n-\delta)}{2} - Q < 0,$$

so that

$$Q > n(n-\delta) + \frac{\delta}{2}(\delta+1) - 1$$

or

$$Q \geqq \left(\frac{m+3}{2}\right)^2 - \frac{\delta^2}{4} + \frac{\delta}{2}(\delta+1),$$

hence

$$Q > \left(\frac{m+3}{2}\right)^2 \text{ unless } \delta = 0.$$

It follows therefore that $z > \frac{m+3}{2}$ could not be the lowest order of a curve through the group, nor $x < z \leqq \frac{m+3}{2}$, for such a minimum group would consist of $z(m+3-z)$ or more than xy points. Since $i < x$ was supposed impossible, c_x is the curve of lowest order through the group. If $Q < xy$ and G_Q primitive, $c_i\,(i < x)$ can be passed through it. If $Q < 2(m+1)$, G_Q lies on a straight line; if $Q < 3m$, G_Q lies on a straight line or a conic.

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[*] Küpper, *l. c.*

13. Consider c_5. We have seen that no g_6^2 lies upon it except that defined by straight lines. If a g_6^2 be possible, it must be a special series, hence each G_6 lies on ϕ_2. Pass a c_2 through a G_6. The residual is a G_4, through which ∞^2 conics should be possible, but this is only possible when all four points are collinear. The ∞^2 lines would then have to cut c_5 in 6 points, but this is impossible if c_5 is irreducible. *A non-singular quintic curve cannot be transformed into a sextic by any birational transformation.* Since $7 = 2n - 3$, the $\infty^2 c_2$ through three arbitrary points on c_5 will define a g_7^2. In exactly the same way it can be shown that a c_6 of $p = 10$ cannot be transformed into a c_7 nor c_8.

14. A g_Q^q is special if $q > Q - p + 1$. Hence if $q = 2$, and $p = \frac{1}{2}(n - 1)(n - 2)$, the group G_Q^2 will be special for every value of $\varkappa < n - 3$ when $Q = \varkappa n - \beta$, $\beta < n$. We need consider only $\varkappa = 2$ and $\beta \geqq 3$, in which case $G_{2n-\beta}^2$ is always defined by ϕ_{n-3}. But these adjoint ϕ_{n-3} are composite, and the variable curve is of order $\varkappa$ or $\varkappa - 1$, hence all the $g_{2n-\beta}^2$ can be cut from c_n by c_2, and hence $\beta = 3$. *In general a non-singular c_n cannot be birationally transformed into a curve of order lower than $2n - 3$. It can always be transformed into a c_{2n-3} by means of quadric inversion, which is birational for the entire plane.*

This method will also apply to singular curves with special moduli, but for complicated P_i the number of particular cases becomes very large.

CORNELL UNIVERSITY, *December*, 1906.

Surfaces with the same Spherical Representation of their Lines of Curvature as Spherical Surfaces.

By Luther Pfahler Eisenhart.

INTRODUCTION.

In several memoirs* we have studied the surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces. It is now our purpose to consider the surfaces with the same representation as spherical surfaces with a view to deriving significant theorems similar to those for A-surfaces,* and also theorems which of necessity have no analogues in the theory of the latter surfaces.

After finding in §1 reduced forms of the equations of Gauss and Codazzi to be satisfied by the fundamental quantities of the surface, we derive the expressions of the latter for the surfaces parallel to a given spherical surface and note that two of them are surfaces of constant mean curvature — as found by Bonnet.† We say that all the surfaces with the same spherical representation form a group; evidently the spherical surface of unit curvature of the group determines the group, and in this sense it may be said to be *associated* with each member of it. Bonnet has shown ‡ that, given one of the above surfaces, there is a unique surface of the same kind applicable to it with correspondence of the lines of curvature, and that these are the only surfaces possessing this property; on this account we call them *surfaces of Bonnet.* It is shown that the two spherical surfaces associated with a pair of applicable surfaces of Bonnet are the Hazzidakis transforms[*] of one another.

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* Surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces, Amer. Journ., vol. 27, pp. 113–172 (1905); we have called them A-surfaces and hereafter this memoir is referred to thus: A. p. 113.

Surfaces analogous to Surfaces of Bianchi, Annali, vol. 12, pp. 113–143 (1905).

† Note sur une propriété de maximum relative à la sphere, Nouv. Annal. de Math., vol. 12, p. 433 (1853); also, Bianchi, Lezioni, II, p. 437.

‡ Mémoire sur la théorie des surfaces applicables sur une surface donnée, Journ. de l'Ecole Polytech., Cahier 42, p. 44 et seq.

Bianchi has established[†] an imaginary transformation of spherical surfaces which is similar to the Bäcklund transformations of pseudospherical surfaces. In § 2 we have given a generalization of this transformation making it applicable to any surface of Bonnet in somewhat the same manner that we did for *A*-surfaces. As in the case of the latter a theorem of permutability can be established so that the knowledge of the general transformation of a surface of Bonnet enables one to find, by algebraic processes, all the transformations of the transforms of the original surface. This is done in § 3.

By means of the theorems of permutability we find in § 4 two imaginary transformations which, when applied successively, transform a given real surface of Bonnet into a new real surface. In particular, we consider the case where the latter belongs to the same group as the original surface.

In § 5 we apply the generalized Bäcklund transformation to two applicable surfaces of Bonnet and show that the functions determining the transformations can be chosen so that after each surface has been subjected to two transformations the resulting surfaces of Bonnet are applicable.

With the aid of the functions giving the generalized Bäcklund transformations we can define a general transformation from a given surface of Bonnet to an imaginary one of the same group, as in the case of *A*-surfaces.[‡] When such a transformation is known, we can find without quadrature another which together with the former transforms the original surface into a real surface of Bonnet of the same group. These results are obtained in § 6. In § 7 several particular solutions are found giving surfaces whose coordinates are expressed in forms similar to those which define the surfaces of Bianchi and the surfaces analogous to them, which we have considered elsewhere.[§]

In § 8 we show that, when one has a surface of Bonnet, *S*, and knows a Bäcklund transformation of it into another surface of Bonnet, then he can find by algebraic processes the unique surface of Bonnet applicable to *S* with correspondence of the lines of curvature.

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[*] Bianchi, Lezioni, II, p. 437.

[†] Bianchi, Lezioni, II, p. 452.

[‡] Surfaces analogous to Surfaces of Bianchi, l. c. p. 116.

[§] Ibid., pp. 118 et seq.

§ 1. *Transformation of Hazzidakis. Theorem of Bonnet.*

When a spherical surface Σ of total curvature $+1$ is referred to its lines of curvature, the parameters can be so chosen that the linear elements of the surface and its spherical representation can be given the respective forms*

$$ds^2 = \sinh^2\omega\, du^2 + \cosh^2\omega\, dv^2, \tag{1}$$

$$ds'^2 = \cosh^2\omega\, du^2 + \sinh^2\omega\, dv^2, \tag{2}$$

where ω is a solution of

$$\frac{\partial^2\omega}{du^2} + \frac{\partial^2\omega}{dv^2} + \sinh\omega\cosh\omega = 0. \tag{3}$$

Denote by S any surface with its lines of curvature represented upon the sphere by the same lines as Σ, and write its linear element thus

$$ds^2 = E\,du^2 + G\,dv^2. \tag{4}$$

The second fundamental quantities have the forms†

$$D = \sqrt{E}\cosh\omega, \qquad D' = 0, \qquad D'' = \sqrt{G}\sinh\omega. \tag{5}$$

The Codazzi and Gauss fundamental equations‡ for S are satisfied, if E and G are such that

$$\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{du} = \frac{\partial\omega}{du}, \qquad \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{dv} = \frac{\partial\omega}{dv}. \tag{6}$$

Surfaces satisfying these conditions will be called *surfaces of Bonnet.* When a system of lines on the sphere leads to a linear element of the form (2), the determination of all the surfaces of Bonnet with this representation of their lines of curvature requires the integration of an equation of Laplace.§ We shall say that these surfaces form a *group,* which evidently is signalized by the spherical surface Σ of the group. It is evident that all of the parallels of a surface of Bonnet are surfaces of the same kind.

On the assumption that E and G are functions of ω alone the equations (6) reduce to forms from which it can be shown that the most general expressions for E and G, on the given hypothesis, are such that

$$\sqrt{E} = c_1\sinh\omega + c_2\cosh\omega, \qquad \sqrt{G} = c_2\sinh\omega + c_1\cosh\omega. \tag{7}$$

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[*] Bianchi, II, p. 436.

[†] Ibid., I, p. 150.

[‡] Ibid., I, p. 122.

[§] cf. *A*, p. 118.

P14730

In particular, when $c_2 = 0$, S is Σ, or homothetic to it, and when $c_1 = 0$, S is a sphere concentric with the unit sphere. When

$$|c_1| = |c_2| = R,$$

we have

$$E = G = R^2 e^{\pm 2\omega}, \tag{8}$$

where the upper or lower sign obtains according as c_1 and c_2 have the same or opposite signs. In like manner the radii of curvature have the expressions

$$\rho_1 = \pm R \frac{e^{\pm\omega}}{\cosh\omega}, \qquad \rho_2 = R \frac{e^{\pm\omega}}{\sinh\omega}, \tag{9}$$

so that the mean curvature of the surfaces is $\pm \frac{1}{R}$. Moreover, it can be shown that, for every surface with constant mean curvature, the parameters of the lines of curvature can be chosen so that the fundamental quantities of the first order are of the form (8) and the principal radii are given by (9), or in inverse order; here ω is any solution of equation (3).

Bonnet showed that surfaces of constant mean curvature are parallel to certain spherical surfaces. But it can be proved readily that all the surfaces of Bonnet with the spherical representation (2) and whose fundamental quantities of the first order are of the form (7), with c_1 and c_2 arbitrary, are parallel to the spherical surface Σ associated with them, or are homothetic of the surfaces parallel to it.*

From the theory of applicability of surfaces we know that there is a double family of lines of the unit sphere for which the parameters can be chosen so that the linear element takes the form (1). If the linear element of surfaces with this representation of their lines of curvature be written in the form

$$ds_1^2 = E_1 du^2 + G_1 dv^2,$$

we may take for the fundamental quantities of the second order

$$D_1 = \sqrt{E_1}\sinh\omega, \qquad D_1' = 0, \qquad D_1'' = \sqrt{G_1}\cosh\omega$$

and the Gauss and Codazzi equations reduce to (3) and

$$\frac{1}{\sqrt{E_1}} \frac{\partial\sqrt{G_1}}{\partial u} = \frac{\partial\omega}{\partial u}, \qquad \frac{1}{\sqrt{G_1}} \frac{\partial\sqrt{E_1}}{\partial v} = \frac{\partial\omega}{\partial v}. \tag{10}$$

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* cf. *A*, p. 124.

In the first place we remark that the latter equations are satisfied by

$$E_1 = \cosh^2 \omega, \qquad G_1 = \sinh^2 \omega.$$

The corresponding surface is seen to be spherical; call it Σ_1. Bianchi has called it the *Hazzidakis transform* of Σ.*

Again, on comparing (6) and (10) it is seen that a solution of equations (10) is given by

$$E_1 = E, \qquad G_1 = G.$$

Hence the theorem:

Every surface in the group of surfaces of Bonnet associated with a spherical surface Σ is applicable to one of the surfaces of Bonnet associated with the Hazzidakis transform of Σ.

It is evident that the lines of curvature correspond on each pair of these applicable surfaces. Bonnet has shown that these are the only pairs of applicable surfaces with this property.

As in the case of the A-surfaces,† it can be shown that:

The spherical surfaces and their parallels are the only surfaces of Bonnet which are Weingarten surfaces.

§ 2. *Generalized Bäcklund Transformations of Surfaces of Bonnet.*

Consider a point M on a surface of Bonnet and in the tangent plane at this point draw a line through M, denoting by θ the angle which it makes with the positive direction of the tangent to the curve $v =$ const. through M. It is our purpose to consider the envelope of the plane which meets the tangent plane, under constant angle σ, in the line as above drawn.

Denote by M_1 the point of contact of the above plane with its envelope. From M_1 drop a perpendicular to the line of intersection of the two planes, and denote its length by μ. Further, let λ denote the distance of the foot of this perpendicular from M.

We refer the surface to the moving rectangular axes formed by the tangents to the lines of curvature at M and the normal to the surface at this point.

The coordinates of M_1 with respect to these axes are

$$\lambda \cos\theta - \mu \cos\sigma \sin\theta, \qquad \lambda \sin\theta + \mu \cos\sigma \cos\theta, \qquad \mu \sin\sigma. \tag{11}$$

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* Lezioni, II, p. 439.

† A, p. 125.

The projections upon these axes of a small displacement of M_1 are found to be*

$$\left.\begin{aligned}
&d(\lambda\cos\theta - \mu\cos\sigma\sin\theta) + \sqrt{E}\,du + \mu\sin\sigma\cosh\omega\,du \\
&\qquad + (\lambda\sin\theta + \mu\cos\sigma\cos\theta)\left(\frac{\partial\omega}{\partial v}du - \frac{\partial\omega}{\partial u}dv\right), \\
&d(\lambda\sin\theta + \mu\cos\sigma\cos\theta) + \sqrt{G}\,dv - (\lambda\cos\theta - \mu\cos\sigma\sin\theta) \\
&\qquad \left(\frac{\partial\omega}{\partial v}du - \frac{\partial\omega}{\partial u}dv\right) + \mu\sin\sigma\sinh\omega\,dv, \\
&\sin\sigma\,d\mu - \cosh\omega\,(\lambda\cos\theta - \mu\cos\sigma\sin\theta)\,du - \sinh\omega\,(\lambda\sin\theta \\
&\qquad + \mu\cos\sigma\cos\theta)\,dv.
\end{aligned}\right\} \quad (12)$$

The calculations which follow are more readily made, if we replace the preceding expressions by the projections of a displacement of M_1 on the line of intersection of the planes (call it MP), the line MQ perpendicular to the latter, lying in the tangent plane, and the normal to the surface. From (12) it follows that these projections are

$$\left.\begin{aligned}
&d\lambda - \mu\cos\sigma\,d\theta + \sqrt{E}\cos\theta\,du + \sqrt{G}\sin\theta\,dv + \left(\frac{\partial\omega}{\partial v}du - \frac{\partial\omega}{\partial u}dv\right)\mu\cos\sigma \\
&\qquad + \mu\sin\sigma\,(\cos\theta\cosh\omega\,du + \sin\theta\sinh\omega\,dv), \\
&\lambda d\theta + \cos\sigma\,d\mu - \sqrt{E}\sin\theta\,du + \sqrt{G}\cos\theta\,dv - \left(\frac{\partial\omega}{\partial v}du - \frac{\partial\omega}{\partial u}dv\right)\lambda \\
&\qquad - \mu\sin\sigma\,(\sin\theta\cosh\omega\,du - \cos\theta\sinh\omega\,dv), \\
&\sin\sigma\,d\mu - \cosh\omega\,(\lambda\cos\theta - \mu\cos\sigma\sin\theta)\,du - \sinh\omega\,(\lambda\sin\theta + \mu\cos\sigma\cos\theta).
\end{aligned}\right\} \quad (13)$$

The direction-cosines of the given plane with respect to the lines MP, MQ, MN are evidently

$$0, \qquad -\sin\sigma, \qquad \cos\sigma. \qquad (14)$$

Since this plane is to be tangent of the locus of the point M_1, the above functions must satisfy the following conditions:

$$\left.\begin{aligned}
\lambda\sin\sigma\left(\frac{\partial\theta}{\partial u} - \frac{\partial\omega}{\partial v}\right) &= \sqrt{E}\sin\theta\sin\sigma - \lambda\cos\sigma\cosh\omega\cos\theta + \mu\sin\theta\cosh\omega, \\
\lambda\sin\sigma\left(\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u}\right) &= -\sqrt{G}\cos\theta\sin\sigma - \lambda\cos\sigma\sinh\omega\sin\theta - \mu\cos\theta\sinh\omega.
\end{aligned}\right\} \quad (15)$$

We shall consider first the case where the surface S is spherical, and inquire whether equations can be satisfied when λ and μ are constant, the latter being zero. In consequence of (1), equations (15) reduce for this case to

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* Darboux, Leçons, II, p. 385.

$$\left.\begin{aligned} \lambda \sin\sigma\left(\frac{\partial\theta}{\partial u} - \frac{\partial\omega}{\partial v}\right) &= \sin\sigma \sin\theta \sinh\omega - \lambda\cos\sigma\cos\theta\cosh\omega, \\ \lambda \sin\sigma\left(\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u}\right) &= -\sin\sigma\cos\theta\cosh\omega - \lambda\cos\sigma\sin\theta\sinh\omega. \end{aligned}\right\} \quad (16)$$

Differentiate the first with respect to v and the second with respect to u, and subtract; since ω is a solution of equation (3) the resulting equation is

$$\lambda^2 = -\sin^2\sigma.$$

There is no loss of generality in replacing this by

$$\lambda = i\sin\sigma.$$

If this value be substituted in (16) and the resulting equations be differentiated with respect to u and v respectively, we have upon adding the equation

$$\frac{\partial^2\theta}{du^2} + \frac{\partial^2\theta}{dv^2} = \sin\theta\cos\theta.$$

If we introduce a new function ω_1 defined by*

$$\theta = \frac{\pi}{2} + i\omega_1, \quad (17)$$

it is found that ω_1 is a solution of equation (3), and equations (16) take the form

$$\left.\begin{aligned} \sin\sigma\left(\frac{\partial\omega_1}{\partial u} + i\frac{\partial\omega}{\partial v}\right) &= -\sinh\omega\cosh\omega_1 + \cos\sigma\cosh\omega\sinh\omega_1, \\ \sin\sigma\left(i\frac{\partial\omega_1}{\partial v} + \frac{\partial\omega}{\partial u}\right) &= \cosh\omega\sinh\omega_1 - \cos\sigma\sinh\omega\cosh\omega_1. \end{aligned}\right\} \quad (18)$$

Now the expressions (13) reduce to

$$\begin{gathered} -i\sinh\omega\sinh\omega_1\, du + \cosh\omega\cosh\omega_1\, dv, \\ -\cos\sigma\,(\cosh\omega\sinh\omega_1\, du + i\sinh\omega\cosh\omega_1\, dv), \\ -\sin\sigma\,(\cosh\omega\sinh\omega_1\, du + i\sinh\omega\cosh\omega_1\, dv), \end{gathered} \quad (13')$$

from which it follows that the linear element of the locus of M_1 is

$$ds_1^2 = \sinh^2\omega_1\, du^2 + \cosh^2\omega_1\, dv^2. \quad (19)$$

In order to prove that the parametric lines on this surface, Σ_1, are the lines of curvature, we make use of a method followed by Darboux† under similar conditions in a study of the Bäcklund transformations of pseudospherical surfaces.

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* cf. Bianchi, Lezioni, II, p. 454,

† Leçons, III, p. 435,

From (14) and (17) it follows that the direction-cosines of the normal to Σ_1 at M_1 with respect to the original moving axes are

$$\sin\sigma\cosh\omega_1, \qquad i\sin\sigma\sinh\omega_1, \qquad \cos\sigma, \tag{20}$$

and consequently the coordinates of a point P on this normal at a constant distance a from M_1 are

$$\sin\sigma\,(\sinh\omega_1 + a\cosh\omega_1), \quad i\sin\sigma\,(\cosh\omega_1 + a\sinh\omega_1), \quad a\cos\sigma.$$

Since ω_1 is a solution of equations (18), the projections upon the original axes of a displacement of the point P are reducible to

$$\begin{aligned}
&-(\sinh\omega\sinh\omega_1 - \cos\sigma\cosh\omega\cosh\omega_1)(\sinh\omega_1 + a\cosh\omega_1)\,du\\
&\qquad - i(\cosh\omega\sinh\omega_1 - \cos\sigma\sinh\omega\cosh\omega_1)(\cosh\omega_1 + a\sinh\omega_1)\,dv,\\
&-i(\sinh\omega\cosh\omega_1 - \cos\sigma\cosh\omega\sinh\omega_1)(\sinh\omega_1 + a\cosh\omega_1)\,du\\
&\qquad + (\cosh\omega\cosh\omega_1 - \cos\sigma\sinh\omega_1\sinh\omega)(\cosh\omega_1 + a\sinh\omega_1)\,dv,\\
&-\cosh\omega\sin\sigma\,(\sinh\omega_1 + a\cosh\omega_1)\,du\\
&\qquad - i\sinh\omega\sin\sigma\,(\cosh\omega_1 + a\sinh\omega_1)\,dv.
\end{aligned}$$

From these expressions it is readily found that the linear element of the locus of P is

$$ds^2 = (\sinh\omega_1 + a\cosh\omega_1)^2\,du^2 + (\cosh\omega_1 + a\sinh\omega_1)^2\,dv^2.$$

As defined this surface is parallel to the locus of M_1. Since the parametric lines form an orthgonal system on both surfaces they are the lines of curvature for these surfaces.

Since ω_1 is a solution of equation (3), the surface Σ_1 with the linear element (19) is a spherical surface, whose spherical representation is given by

$$ds_1'^2 = \cosh^2\omega_1\,du^2 + \sinh^2\omega_1\,dv^2. \tag{21}$$

As in the case of the Bäcklund transformations of pseudospherical surfaces, equations (18) can be transformed to the Riccati type, so that for a given value of σ the general integral contains an arbitrary constant. It is evident that these transforms, doubly-infinite in number, are imaginary.*

We pass now to the consideration of the case where the surface S is any surface of Bonnet, and discuss the case when ω_1 is any solution of equations (18). Equations (15) reduce to

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* cf. Bianchi, Lezioni, II, p. 454.

$$\begin{aligned} i\sqrt{E}\sin\sigma &= \lambda\sinh\omega - i\mu\cosh\omega, \\ i\sqrt{G}\sin\sigma &= \lambda\cosh\omega - i\mu\sinh\omega, \end{aligned} \tag{22}$$

from which we get

$$\begin{aligned} \lambda &= i\sin\sigma\,(-\sqrt{E}\sinh\omega + \sqrt{G}\cosh\omega), \\ \mu &= \sin\sigma\,(-\sqrt{E}\cosh\omega + \sqrt{G}\sinh\omega), \end{aligned} \tag{23}$$

If these values for λ and μ be substituted in (13), it is found that the projections of a displacement of M_1 are

$$\begin{gathered} -i\sinh\omega\sqrt{E_1}\,du + \cosh\omega\sqrt{G_1}\,dv, \\ -\cos\sigma\,(\cosh\omega\sqrt{E_1}\,du + i\sinh\omega\sqrt{G_1}\,dv), \\ -\sin\sigma\,(\cosh\omega\sqrt{E_1}\,du + i\sinh\omega\sqrt{G_1}\,dv), \end{gathered} \tag{24}$$

where we have put

$$\begin{aligned} \sqrt{E_1} &= \sin\sigma\left(\frac{\partial\sqrt{E}}{\partial u} - \sqrt{G}\,\frac{\partial\omega}{\partial u}\right) - \frac{i\lambda\sinh\omega_1 + \mu\cos\sigma\cosh\omega_1}{\sin\sigma}, \\ \sqrt{G_1} &= i\sin\sigma\left(\frac{\partial\sqrt{G}}{\partial v} - \sqrt{E}\,\frac{\partial\omega}{\partial v}\right) - \frac{i\lambda\cosh\omega_1 + \mu\cos\sigma\sinh\omega_1}{\sin\sigma}. \end{aligned} \tag{25}$$

From (24) we find that the linear element of the transform is

$$ds_1^2 = E_1\,du^2 + G_1\,dv^2.$$

The tangent plane to this surface at a point M_1 is evidently parallel to the tangent plane to the surface Σ_1, which is the spherical transform, by means of the same σ and ω_1 of the surface Σ associated with the original S; corresponding points on Σ_1 and S_1 being the transforms of the points on Σ and S with the same spherical representation. Hence the spherical representation of S_1 is given by (21), from which it follows that the parametric lines on S_1 are its lines of curvature and consequently S_1 is a surface of Bonnet. Therefore, each solution of equations (18) gives a transformation of the surfaces of Bonnet with the spherical representation (2) into a group with the representation (21).

From (23) it is seen that for a given σ the points M_1, on all the transforms of a surface of Bonnet, corresponding to a point M on the latter lie on an imaginary circle whose axis is the normal to S at M.

§ 3. *Theorem of Permutability.*

It is now our purpose to show that there exists for surfaces of Bonnet a theorem of permutability similar to the one which we established for A-surfaces.* Thus, it will be shown that if a given surface S be transformed by means of (ω_1, σ_1) and (ω_2, σ_2) into the surfaces S_1 and S_2 respectively, there can be found without quadratures a function ω_3 such that S_1 and S_2 are transformed into the same surface S_3 by means of (ω_3, σ_2) and (ω_3, σ_1) respectively.

Denote by λ_1, μ_1 the lengths determining the point M_1 on S_1 corresponding to M on S, and by λ_{13}, μ_{13} the similar functions giving the transformation from M_1 to M_3. From (23) it is seen that these functions are of the form

$$\left.\begin{aligned} \lambda_1 &= i \sin\sigma_1 \left(-\sqrt{E}\sinh\omega + \sqrt{G}\cosh\omega\right), \\ \mu_1 &= \sin\sigma_1 \left(-\sqrt{E}\cosh\omega + \sqrt{G}\sinh\omega\right), \\ \lambda_{13} &= i \sin\sigma_2 \left(-\sqrt{E_1}\sinh\omega_1 + \sqrt{G_1}\cosh\omega_1\right), \\ \mu_{13} &= \sin\sigma_2 \left(-\sqrt{E_1}\cosh\omega_1 + \sqrt{G_1}\sinh\omega_1\right). \end{aligned}\right\} \quad (27)$$

Denote by θ_3 the angle formed with the tangent to the line $v =$ const. through M_1 by the line of intersection of the tangent planes to S_1 and S_3. The projections, on the trihedron formed by the normal to S_1 and the tangents to the lines of curvature at M_1, of the line $M_1 M_3$ are

$$\mu_{13}\sin\sigma_2, \quad \lambda_{13}\cos\theta_3 - \mu_{13}\cos\sigma_2\sin\theta_3, \quad \lambda_{13}\sin\theta_3 + \mu_{13}\cos\sigma_2\cos\theta_3.$$

It is evident that this trihedron is parallel to the similar trihedron for the transform Σ_1 of the spherical surface Σ. Hence it follows from (13′) and (14) that the direction-cosines of the angles which the axes of the above trihedron make with the lines MP, MQ, MN for S are

$$\begin{array}{lll} -i\sinh\omega, & -\cos\sigma_1\cosh\omega, & -\sin\sigma_1\cosh\omega, \\ \cosh\omega, & -i\cos\sigma_1\sinh\omega, & -i\sin\sigma_1\sinh\omega, \\ 0, & -\sin\sigma_1, & \cos\sigma_1. \end{array}$$

Hence, if θ_3 be replaced by $\frac{\pi}{2} + i\omega_3$, the coordinates of M_3 with respect to the axes MP, MQ, MN are

$$\begin{gathered} \lambda_1 + \lambda_{13}\cosh(\omega_3 - \omega) - i\mu_{13}\cos\sigma_2\sinh(\omega_3 - \omega), \\ \mu_1\cos\sigma_1 + \cos\sigma_1\left[i\lambda_{13}\sinh(\omega_3 - \omega) + \mu_{13}\cos\sigma_2\cosh(\omega_3 - \omega)\right] - \mu_{13}\sin\sigma_1\sin\sigma_2, \\ \mu_1\sin\sigma_1 + \sin\sigma_1\left[i\lambda_{13}\sinh(\omega_3 - \omega) + \mu_{13}\cos\sigma_2\cosh(\omega_3 - \omega)\right] + \mu_{13}\cos\sigma_1\sin\sigma_2. \end{gathered}$$

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* A, p. 154.

From these it is readily found that the coordinates x_3, y_3, z_3 of M_3 with respect to the axes at M formed by the tangents to the lines of curvature and the normal are

$$\left.\begin{aligned}
x_3 = & -i\lambda_1 \sinh\omega_1 - \mu_1 \cos\sigma_1 \cosh\omega_1 - (i\lambda_{13} \sinh\omega_1 + \mu_{13} \cos\sigma_1 \cos\sigma_2 \cosh\omega_1) \\
& \cosh(\omega_3 - \omega) - (i\lambda_{13} \cos\sigma_1 \cosh\omega_1 + \mu_{13} \cos\sigma_2 \sinh\omega_1) \sinh(\omega_3 - \omega) \\
& + \mu_{13} \sin\sigma_1 \sin\sigma_2 \cosh\omega_1, \\
y_3 = & \lambda_1 \cosh\omega_1 - i\mu_1 \cos\sigma_1 \sinh\omega_1 + (\lambda_{13} \cosh\omega_1 - i\mu_{13} \cos\sigma_1 \cos\sigma_2 \sinh\omega_1) \\
& \cosh(\omega_3 - \omega) + (\lambda_{13} \cos\sigma_1 \sinh\omega_1 - i\mu_{13} \cos\sigma_2 \cosh\omega_1) \sinh(\omega_3 - \omega) \\
& + i\mu_{13} \sin\sigma_1 \sin\sigma_2 \sinh\omega_1, \\
z_3 = & \mu_1 \sin\sigma_1 + \sin\sigma_1 [i\lambda_{13} \sinh(\omega_3 - \omega) + \mu_{13} \cos\sigma_2 \cosh(\omega_3 - \omega)] \\
& + \mu_{13} \cos\sigma_1 \sin\sigma_2 .
\end{aligned}\right\} \quad (28)$$

According to the statement of our problem, it must be shown that S_2 is transformed by means of the same ω_3 and σ_1, instead of σ_2, into the surface S_3 defined by (28). For the moment we denote the new transform by S_3' and its coordinates by x_3', y_3', z_3'. It is clear that the expressions for the latter are given by (28), if the subscripts 1 and 2 are interchanged and the subscript 13 is replaced by 23.

In order that the two surfaces coincide we must have

$$\begin{gathered}
-i \sinh\omega_1 (x_3' - x_3) + \cosh\omega_1 (y_3 - y_3') = 0, \\
-i \sinh\omega_2 (x_3' - x_3) + \cosh\omega_2 (y_3 - y_3') = 0, \\
z_3' = z_3 .
\end{gathered}$$

By substitution the latter become

$$\left.\begin{aligned}
& [\lambda_{23} \cosh(\omega_2 - \omega_1) - i\mu_{23} \cos\sigma_1 \cos\sigma_2 \sinh(\omega_2 - \omega_1) - \lambda_{13}] \cosh(\omega_3 - \omega) \\
& \quad + i[\mu_{13} \cos\sigma_2 - \mu_{23} \cos\sigma_1 \cosh(\omega_2 - \omega_1) - i\lambda_{23} \cos\sigma_2 \sinh(\omega_2 - \omega_1)] \\
& \quad \sinh(\omega_3 - \omega) = \lambda_1 - \lambda_2 \cosh(\omega_2 - \omega_1) + i\mu_2 \cos\sigma_2 \sinh(\omega_2 - \omega_1) \\
& \quad - i\mu_{23} \sin\sigma_1 \sin\sigma_2 \sinh(\omega_2 - \omega_1), \\
& [\lambda_{13} \cosh(\omega_2 - \omega_1) + i\mu_{13} \cos\sigma_1 \cos\sigma_2 \sinh(\omega_2 - \omega_1) - \lambda_{23}] \cosh(\omega_3 - \omega) \\
& \quad + i[\mu_{23} \cos\sigma_1 - \mu_{13} \cos\sigma_2 \cosh(\omega_2 - \omega_1) + i\lambda_{13} \cos\sigma_1 \sinh(\omega_2 - \omega_1)] \\
& \quad \sinh(\omega_3 - \omega) = \lambda_2 - \lambda_1 \cosh(\omega_2 - \omega_1) - i\mu_1 \cos\sigma_1 \sinh(\omega_2 - \omega_1) \\
& \quad + i\mu_{13} \sin\sigma_1 \sin\sigma_2 \sinh(\omega_2 - \omega_1), \\
& (\mu_{13} \sin\sigma_1 \cos\sigma_2 - \mu_{23} \sin\sigma_2 \cos\sigma_1) \cosh(\omega_3 - \omega) + i(\lambda_{13} \sin\sigma_1 - \lambda_{23} \sin\sigma_2) \\
& \quad \sinh(\omega_3 - \omega) = \sin\sigma_2 (\mu_2 - \mu_{13} \cos\sigma_1) - \sin\sigma_1 (\mu_1 - \mu_{23} \cos\sigma_2).
\end{aligned}\right\} \quad (29)$$

We consider first the case where S is a spherical surface; now

$$\lambda_1 = \lambda_{23} = i \sin \sigma_1, \quad \lambda_2 = \lambda_{13} = i \sin \sigma_2, \quad \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0,$$

and the above equations reduce to

$$\begin{aligned}
&[\sin\sigma_1 \cosh(\omega_2-\omega_1) - \sin\sigma_2] \cosh(\omega_3-\omega) + \sin\sigma_1 \cos\sigma_2 \sinh(\omega_2-\omega_1) \sinh(\omega_3-\omega) \\
&\qquad\qquad = \sin\sigma_1 - \sin\sigma_2 \cosh(\omega_2-\omega_1), \\
&[\sin\sigma_2 \cosh(\omega_2-\omega_1) - \sin\sigma_1] \cosh(\omega_3-\omega) - \sin\sigma_2 \cos\sigma_1 \sinh(\omega_2-\omega_1) \sinh(\omega_3-\omega) \\
&\qquad\qquad = \sin\sigma_2 - \sin\sigma_1 \cosh(\omega_2-\omega_1).
\end{aligned}$$

Solving these equations for $\cosh(\omega_3-\omega)$ and $\sinh(\omega_3-\omega)$, we get

$$\left.\begin{aligned}
\cosh(\omega_3-\omega) &= \frac{\sin\sigma_1 \sin\sigma_2 + (\cos\sigma_1\cos\sigma_2 - 1)\cosh(\omega_2-\omega_1)}{\sin\sigma_1\sin\sigma_2\cosh(\omega_2-\omega_1) + \cos\sigma_1\cos\sigma_2 - 1}, \\
\sinh(\omega_3-\omega) &= \frac{(\cos\sigma_2 - \cos\sigma_1)\sinh(\omega_2-\omega_1)}{\sin\sigma_1\sin\sigma_2\cosh(\omega_2-\omega_1) + \cos\sigma_1\cos\sigma_2 - 1}.
\end{aligned}\right\} \tag{30}$$

Since these expressions satisfy the general relation $\cosh^2\alpha - \sinh^2\alpha = 1$, they may be replaced by

$$\tanh\left(\frac{\omega_3-\omega}{2}\right) = \frac{\sin\left(\frac{\sigma_1+\sigma_2}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma_2}{2}\right)} \tanh\left(\frac{\omega_2-\omega_1}{2}\right). \tag{31}$$

It remains for us to show that the function ω_3 thus given satisfies the conditions of the problem. The functions ω_1 and ω_2 must satisfy equations (18) in which σ has the respective values σ_1 and σ_2. In like manner ω_3 must satisfy

$$\left.\begin{aligned}
\sin\sigma_2 \left(\frac{\partial\omega_3}{\partial u} + i\,\frac{\partial\omega_1}{\partial v}\right) &= -\sinh\omega_1\cosh\omega_3 + \cos\sigma_2\cosh\omega_1\sinh\omega_3, \\
\sin\sigma_2 \left(i\,\frac{\partial\omega_3}{\partial v} + \frac{\partial\omega_1}{\partial u}\right) &= \cosh\omega_1\sinh\omega_3 - \cos\sigma_2\sinh\omega_1\cosh\omega_3;
\end{aligned}\right\} \tag{32}$$

and

$$\left.\begin{aligned}
\sin\sigma_1 \left(\frac{\partial\omega_3}{\partial u} + i\,\frac{\partial\omega_2}{\partial v}\right) &= -\sinh\omega_2\cosh\omega_3 + \cos\sigma_1\cosh\omega_2\sinh\omega_3, \\
\sin\sigma_1 \left(i\,\frac{\partial\omega_3}{\partial v} + \frac{\partial\omega_2}{\partial u}\right) &= \cosh\omega_2\sinh\omega_3 - \cos\sigma_1\sinh\omega_2\cosh\omega_3.
\end{aligned}\right\} \tag{33}$$

It is readily found that, when ω_1 and ω_2 are any solutions whatever of equations (18), the function ω_3 given directly by (31) satisfies (32) and (33).*

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* cf. Bianchi, Lezioni, II, p. 458.

Furthermore, if the values of $\sinh(\omega_3-\omega)$ and $\cosh(\omega_3-\omega)$, given by (30), are substituted in (29) together with the expressions (27) for λ and μ, it is found that these conditions are satisfied. Hence we have this theorem:

When two particular transformations of a surface of Bonnet are known, a transformation of the resulting surfaces can be effected by algebraic processes and in each case it gives the same surface.

Consequently, as in the case of *A*-surfaces, when one knows the general transformation of a surface of Bonnet, its transforms can be transformed by algebraic processes.

§ 4. *Real Transformations.*

From the preceding discussion it is clear that all the transforms of S, such as S_1 and S_2, are imaginary; and, in general, the transforms of the latter are imaginary. We seek now surfaces of the latter class which are real.

Denoting by $\bar{\omega}_1$, $\bar{\sigma}_1$ the conjugate-imaginaries of ω_1, σ_1, we put

$$\omega_2 = i\pi - \bar{\omega}_1, \qquad \sigma_2 = \pi - \bar{\sigma}_1. \tag{34}$$

It is found that ω_2 is a solution of equations (18) with σ given by (34), provided that ω_1 is a solution of these equations with σ_1 in place of σ.

If we put for brevity

$$\begin{aligned} a &= -\sqrt{E}\sinh\omega + \sqrt{G}\cosh\omega, & b &= -\sqrt{E}\cosh\omega + \sqrt{G}\sinh\omega, \\ c &= -\sin\sigma_1\sin\sigma_2\left(\frac{\partial\sqrt{E}}{\partial u} - \sqrt{G}\frac{\partial\omega}{\partial u}\right), & d &= \sin\sigma_1\sin\sigma_2\left(\frac{\partial\sqrt{G}}{\partial v} - \sqrt{E}\frac{\partial\omega}{\partial v}\right), \end{aligned} \tag{35}$$

the expressions (27) may be written thus

$$\begin{aligned} \lambda_1 &= i\sin\sigma_1\, a, \qquad \mu_1 = \sin\sigma_1\, b, \\ \lambda_{13} &= i\,(c\sinh\omega_1 + i\,d\cosh\omega_1) + i\sin\sigma_2\, a, \\ \mu_{13} &= c\cosh\omega_1 + i\,d\sinh\omega_1 + \cos\sigma_1\sin\sigma_2\, b. \end{aligned} \tag{36}$$

Since $\sin\sigma_1$ and $\sin\sigma_2$ are conjugate-imaginaries and the other functions in (35) pertain to S, the functions a, b, c, d are real.

When the above values are substituted in the expression (28) for x_3, and we make use of (30), we get

$$\left.\begin{aligned} x_3 = \frac{1}{D}\{ &-a\,(\cos\bar{\sigma}_1 + \cos\sigma_1)(\sin\bar{\sigma}_1\cos\sigma_1\sinh\bar{\omega}_1 + \sin\sigma_1\cos\bar{\sigma}_1\sinh\omega_1) \\ &+ b\cos\sigma_1\cos\bar{\sigma}_1(\cos\bar{\sigma}_1 + \cos\sigma_1)(\sin\bar{\sigma}_1\cosh\bar{\omega}_1 + \sin\sigma_1\cosh\omega_1) \\ &+ c\,[(\cos\bar{\sigma}_1 + \cos\sigma_1)^2\cosh\omega_1\cosh\bar{\omega}_1 - \sin\sigma_1\sin\bar{\sigma}_1 \\ &\qquad\qquad - (\cos\sigma_1\cos\bar{\sigma}_1 + 1)\cosh(\bar{\omega}_1 + \omega_1) \\ &+ i\,d\,[(\cos\bar{\sigma}_1 + \cos\sigma_1)(\cos\bar{\sigma}_1\sinh\omega_1\cosh\bar{\omega}_1 - \cos\sigma_1\sinh\bar{\omega}_1\cosh\omega_1)]\}, \end{aligned}\right\} \tag{37}$$

where D denotes the denominator in (30). Since the above expression for x_3 is real and similar results follow for y_3 and z_3, it is evident that S_3 is a real surface.

For the values (34) equation (31) becomes

$$\tanh\left(\frac{\omega_3-\omega}{2}\right)=\frac{\cos\left(\frac{\sigma_1-\bar{\sigma}_1}{2}\right)}{\cos\left(\frac{\sigma_1+\bar{\sigma}_1}{2}\right)}\coth\left(\frac{\omega_1+\bar{\omega}_1}{2}\right),\qquad\Bigg\}\;(38)$$

from which it is seen that ω_3 is real.

Returning to the general case, we remark that when $\sigma_2=\sigma_1$, we get from (31)

$$\omega_3-\omega=(2m+1)i\pi. \tag{39}$$

Moreover, if this value of ω_3 be substituted in (32) and (33), they reduce to (18). Now the linear element of the spherical representation of S_3, namely

$$ds_3'^2=\cosh^2\omega_3\,du^2+\sinh^2\omega_3\,dv^2$$

reduces to (2). Hence S_3 belongs to the same group as S; it is the envelope of the plane containing the points M_1, M_2, &c., which are the transforms of M by means of the general solution ω_1 of equations (18) in which $\sigma=\sigma_1$. We will consider, in particular, the case where S_3 is real.

Referring to (38), we see that if ω_3, given by (39), be a solution for any function ω_1, $\sigma_1+\bar{\sigma}_1$ is an odd multiple of π. Without loss of generality we may take

$$\sigma_1+\bar{\sigma}_1=\pi.$$

Now σ_1 and $\bar{\sigma}_1$ are of the form

$$\sigma_1=\frac{\pi}{2}+i\tau,\qquad \bar{\sigma}_1=\frac{\pi}{2}-i\tau,$$

hence

$$\sin\sigma_1=\sin\sigma_2=\cosh\tau,\qquad \cos\sigma_1=\cos\sigma_2=-i\sinh\tau. \tag{40}$$

For these values the expressions (28) for the projections upon the original trihedron of the length MM_3 reduce to

$$c,\qquad d,\qquad b\sin^2\sigma, \tag{41}$$

in consequence of (36).

With respect to axes fixed in space the direction-cosines of the tangents to the lines of curvature of Σ, and consequently of S, will be denoted by X_1, Y_1, Z_1; X_2, Y_2, Z_2. Hence if we denote by (x, y, z) and (x', y', z') the coordinates, with respect to these axes, of corresponding points on S and S_3, we have from (35), (40) and (41),

$$\left.\begin{aligned} x' = x - \cosh^2\tau \left[\left(\frac{\partial\sqrt{E}}{\partial u} - \sqrt{G}\frac{\partial\omega}{\partial u}\right)X_1 - \left(\frac{\partial\sqrt{G}}{\partial v} + \sqrt{E}\frac{\partial\omega}{\partial v}\right)X_2\right] \\ + \left(\sqrt{E}\cosh\omega - \sqrt{G}\sinh\omega\right)X \end{aligned}\right\} \quad (42)$$

and similar expressions for y' and z'. It is readily shown that these define a parallel to S, when the latter is a spherical surface or one of its parallels, and only in this case.

§ 5. *Bäcklund Transformations of Applicable Surfaces of Bonnet.*

We pass now to the consideration of the transformations of the surfaces of Bonnet whose spherical representation is given by (1). The associated spherical surface, Σ', is the Hazzidakis transform of Σ, and its linear element is given by (2).

For this case the equations analogous to (16) are

$$\lambda' \sin\sigma' \left(\frac{\partial\theta'}{\partial u} - \frac{\partial\omega}{\partial v}\right) = \sin\sigma' \sin\theta' \cosh\omega - \lambda' \cos\sigma' \sinh\omega \cos\theta',$$

$$\lambda' \sin\sigma' \left(\frac{\partial\theta'}{\partial v} + \frac{\partial\omega}{\partial u}\right) = -\sin\sigma' \cos\theta' \sinh\omega - \lambda' \cos\sigma' \cosh\omega \sin\theta'.$$

The conditions of integrability of these equations reduce to

$$\lambda' = i \sin\sigma'$$

and

$$\frac{\partial^2\theta'}{\partial u^2} + \frac{\partial^2\theta'}{\partial v^2} + \sin\theta' \cos\theta' = 0.$$

If we put

$$\theta' = \pi + i\omega_1',$$

this becomes

$$\frac{\partial^2\omega_1'}{\partial u^2} + \frac{\partial^2\omega_1'}{\partial v^2} + \sinh\omega_1' \cosh\omega_1' = 0,$$

and the above equations are reducible to

$$\left.\begin{aligned} \sin\sigma' \left(\frac{\partial\omega_1'}{\partial u} + i\frac{\partial\omega}{\partial v}\right) &= i \sinh\omega_1' \cosh\omega - i\cos\sigma' \cosh\omega_1' \sinh\omega, \\ \sin\sigma' \left(i\frac{\partial\omega_1'}{\partial v} + \frac{\partial\omega}{\partial u}\right) &= -i\cosh\omega_1' \sinh\omega + i\cos\sigma' \sinh\omega_1' \cosh\omega. \end{aligned}\right\} \quad (43)$$

When these equations are compared with (18), it is seen that if σ' be given by

$$\sin\sigma' = i\tan\sigma, \qquad \cos\sigma' = \sec\sigma, \quad (44)$$

the function ω_1 is a solution of equations (43). The linear element of the transform of Σ_1' by means of ω_1 and σ' is

$$ds_1'^2 = \cosh^2\omega_1\, du^2 + \sinh^2\omega_1\, dv^2$$

and the linear element of its spherical representation is

$$ds_1''^2 = \sinh^2\omega_1\, du^2 + \cosh^2\omega_1\, dv^2. \tag{45}$$

From these expressions it is seen that the new surface Σ_1' is the Hazzidakis transform of Σ_1.*

We have seen that the surfaces of Bonnet associated with Σ and those associated with its Hazzidakis transform can be arranged in pairs of applicable surfaces. We shall consider the effect of the preceding transformations on such a pair, S and S'.

Let the linear element of S' be (4) and of its spherical representation (1). From (15) and (43) it is seen that, if we denote by λ' and μ' the functions for S' analogous to λ and μ for S, they are given by

$$\left.\begin{aligned} \lambda' &= \tan\sigma\,(-\sqrt{E}\cosh\omega + \sqrt{G}\sinh\omega), \\ \mu' &= i\tan\sigma\,(\sqrt{E}\sinh\omega - \sqrt{G}\cosh\omega); \end{aligned}\right\} \tag{46}$$

in these expressions, as first found, $\sin\sigma'$ has been replaced by $i\tan\sigma$. It is readily found that the linear element of the transform S_1' is

$$ds_1'^2 = E_1'\, du^2 + G_1'\, dv^2,$$

where

$$\left.\begin{aligned} \sqrt{E_1'} &= \tan\sigma\left(\frac{\partial\sqrt{E}}{\partial u} - \sqrt{G}\,\frac{\partial\omega}{\partial u}\right) - \frac{\lambda'\cosh\omega_1}{\tan\sigma} + i\,\mu'\,\frac{\sinh\omega_1}{\sin\sigma}, \\ \sqrt{G_1'} &= i\tan\sigma\left(\frac{\partial\sqrt{G}}{\partial v} - \sqrt{E}\,\frac{\partial\omega}{\partial v}\right) - \frac{\lambda'\sinh\omega_1}{\tan\sigma} + i\,\mu'\,\frac{\cosh\omega_1}{\sin\sigma}, \end{aligned}\right\} \tag{47}$$

and the spherical representation is given by (45).

A comparison of (23) and (46) shows that

$$\lambda' = \sec\sigma\,.\,\mu, \qquad \mu' = -\sec\sigma\,.\,\lambda.$$

If these values be substituted in (47) and the result be compared with (25), it is found that

$$\sqrt{E_1'} = \sec\sigma\,\sqrt{E_1}, \qquad \sqrt{G_1'} = \sec\sigma\,\sqrt{G_1}. \tag{48}$$

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* cf. Bianchi, Lezioni, II, p. 469.

From this it is seen that a homothetic transformation applied to S_1' will give a surface of Bonnet applicable to S_1. Hence, if S and S' are two applicable surfaces of Bonnet, and S_1 is the Bäcklund transform of S by means of ω_1 and σ, the Bäcklund transform of S' by means of ω_1 and σ', the latter given by (44), is homothetic to the surface applicable to S_1 with preservation of lines of curvature. All of these surfaces are imaginary, but we shall find real ones in consequence of the theorem of permutability.

As before, we denote by S_3 the real surface, which is the transform of S_1 by means of ω_3 and σ_2, these functions being given by (38) and (34); and we write the linear element of S_3 in the form

$$ds_3^2 = E_3\, du^2 + G_3\, dv^2.$$

The preceding results show us that S_1' can be transformed into a surface S_3' by means of ω_3 and σ_2', where σ_2' is defined by

$$\sin \sigma_2' = i \tan \sigma_2, \qquad \cos \sigma_2' = \sec \sigma_2, \tag{49}$$

and S_3' has the same spherical representation as S_3. If the linear element of S_3' be written thus

$$ds_3'^2 = E_3'\, du^2 + G_3'\, dv^2,$$

the functions $\sqrt{E_3}$, $\sqrt{G_3}$; $\sqrt{E_3'}$, $\sqrt{G_3'}$ will have forms similar to (25) and (47). Since they are linear and homogeneous in $\sqrt{E_1}$, $\sqrt{G_1}$; $\sqrt{E_1'}$, $\sqrt{G_1'}$, it follows from (48) that

$$\sqrt{E_3'} = \sec \sigma \sec \sigma_2 \sqrt{E_3}, \qquad \sqrt{G_3'} = \sec \sigma \sec \sigma_2 \sqrt{G_3}. \tag{50}$$

From (34) it follows that in order that S_3' be real we must have

$$\sigma_2' = \pi - \bar{\sigma}'. \tag{51}$$

In consequence of (34) equations (49) may be written

$$\sin \sigma_2' = -i \tan \bar{\sigma}, \qquad \cos \sigma_2' = -\sec \bar{\sigma}$$

and from (44) it follows that

$$\sin \bar{\sigma}' = -i \tan \bar{\sigma}, \qquad \cos \bar{\sigma}' = \sec \bar{\sigma}.$$

Comparing these two sets of equations, we see that condition (51) is satisfied.

In consequence of (34) equations (50) become

$$\sqrt{E_3'} = -\sec\sigma\sec\bar{\sigma}\sqrt{E_3}, \qquad \sqrt{G_3'} = -\sec\sigma\sec\bar{\sigma}\sqrt{G_3}.$$

If we put

$$\sigma = \alpha + i\beta,$$

we find that

$$\sec\sigma\sec\bar{\sigma} = 1,$$

if

$$\sin^2\alpha = \sinh^2\beta, \tag{52}$$

and only in this case. Hence, given two applicable surfaces of Bonnet; by two imaginary transformations of Bäcklund we can obtain a second pair of applicable surfaces of Bonnet. Since α or β is arbitrary and there is an arbitrary constant in the solution ω_1 of equations (18), there is a double infinity of these transformations.

§ 6. *General Determination of Surfaces of Bonnet.*

In the tangent plane to a surface of Bonnet, S, at a point M we draw a line through the point of contact and indicate by θ the angle which it makes with the tangent to the line of curvature $v =$ const. At a point P of this line we draw in the tangent plane the segment PQ of the line perpendicular to PM. In the plane through PQ and normal to PM we draw a segment QR making an angle σ with QP. For convenience we indicate by p, ρ, r the respective lengths MP, PQ, QR. If θ is defined by (17) and (18) the projections, on the trihedron formed by the tangents to the lines of curvature and the normal to S_1 of the segment MR are

$$-[ip\sinh\omega_1 + (\rho + r\cos\sigma)\cosh\omega_1], \; [p\cosh\omega_1 - i(\rho + r\cos\sigma)\sinh\omega_1], \; r\sin\sigma. \tag{53}$$

From these it follows that the projections of a displacement of R are of the form*

$$\left.\begin{aligned}
&-d[ip\sinh\omega_1 + (\rho + r\cos\sigma)\cosh\omega_1] + \sqrt{E}\,du + r\sin\sigma\cosh\omega\,du\\
&\qquad + [p\cosh\omega_1 - i(\rho + r\cos\sigma)\sinh\omega_1]\left(\frac{\partial\omega}{\partial v}du - \frac{\partial\omega}{\partial u}dv\right);\\
&d[p\cosh\omega_1 - i(\rho + r\cos\sigma)\sinh\omega_1] + \sqrt{G}\,dv + r\sin\sigma\sinh\omega\,dv\\
&\qquad + [ip\sinh\omega_1 + (\rho + r\cos\sigma)\cosh\omega_1]\left(\frac{\partial\omega}{\partial v}du - \frac{\partial\omega}{\partial u}dv\right),\\
&\sin\sigma\,dr + \cosh\omega\,[ip\sinh\omega_1 + (\rho + r\cos\sigma)\cosh\omega_1]\,du\\
&\qquad - \sinh\omega\,[p\cosh\omega_1 - (\rho + r\cos\sigma)\sinh\omega_1]\,dv.
\end{aligned}\right\} \tag{54}$$

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* Darboux, Leçons, vol. II, p. 385.

From these it is found that the necessary and sufficient condition that the locus of R be a surface of Bonnet with the same spherical representation of its lines of curvature as S is that p, ρ and r satisfy the following equations

$$\left.\begin{aligned}
&\sin\sigma\cosh\omega_1\frac{\partial p}{\partial u}-i\sin\sigma\sinh\omega_1\frac{\partial\rho}{\partial u}-[p\sinh\omega_1-i(\rho\\
&\qquad\qquad+r\cos\sigma)\cosh\omega_1]\sinh\omega\cosh\omega_1=0,\\
&i\sin\sigma\sinh\omega_1\frac{\partial p}{\partial v}+\sin\sigma\cosh\omega_1\frac{\partial\rho}{\partial v}+i[p\cosh\omega_1-i(\rho\\
&\qquad\qquad+r\cos\sigma)\sinh\omega_1]\cosh\omega\sinh\omega_1=0,\\
&\sin\sigma\frac{\partial r}{\partial u}+[i p\sinh\omega_1+(\rho+r\cos\sigma)\cosh\omega_1]\cosh\omega=0,\\
&\sin\sigma\frac{\partial r}{\partial v}-[p\cosh\omega_1-i(\rho+r\cos\sigma)\sinh\omega_1]\sinh\omega=0.
\end{aligned}\right\}\quad(55)$$

From (54) one finds that the coefficients of the linear element of the new surface are given by

$$\left.\begin{aligned}
\sqrt{E'}&=\sqrt{E}-\left(i\sinh\omega_1\frac{\partial p}{\partial u}+\cosh\omega_1\frac{\partial\rho}{\partial u}\right)+\frac{(\rho\cos\sigma+r)\cosh\omega}{\sin\sigma}\\
&\quad+\frac{i p\sinh\omega\cosh^2\omega_1+(\rho+r\cos\sigma)\sinh\omega_1\cosh\omega_1\sinh\omega}{\sin\sigma},\\
\sqrt{G'}&=\sqrt{G}+\left(\cosh\omega_1\frac{\partial p}{\partial v}-i\sinh\omega_1\frac{\partial\rho}{\partial v}\right)+\frac{i(\rho\cos\sigma+r)\sinh\omega}{\sin\sigma}\\
&\quad+\frac{p\sinh^2\omega_1\cosh\omega-(\rho+r\cos\sigma)i\sinh\omega_1\cosh\omega_1\cosh\omega}{\sin\sigma}.
\end{aligned}\right\}\quad(56)$$

As defined, S' is imaginary, but we shall be able to effect a similar transformation on S' and get a real surface S''.

We have seen that, if ω_1 and σ be replaced by $i\pi-\bar{\omega}_1$ and $\pi-\bar{\sigma}$, equations (18) are satisfied. Moreover, it can be shown that, if equations (55) are satisfied by

$$\sigma,\quad \omega_1,\quad p,\quad \rho,\quad r, \qquad (57)$$

these equations are satisfied also by

$$\pi-\bar{\sigma},\quad i\pi-\bar{\omega}_1,\quad -\bar{p},\quad -\bar{\rho},\quad \bar{r}, \qquad (58)$$

where the bar indicates the conjugate imaginary function.

The successive application of these transformations upon S gives a surface S'', whose coordinates are of the form

$$\left.\begin{aligned} x'' = x - [i\,(p \sinh \omega_1 - \bar{p} \sinh \bar{\omega}_1) + (\rho \cosh \omega_1 + \bar{\rho} \cosh \bar{\omega}_1) \\ + (r \cos \sigma \cosh \omega_1 + \bar{r} \cos \bar{\sigma} \cosh \bar{\omega}_1)]\, X_1 \\ + [(p \cosh \omega_1 + \bar{p} \cosh \bar{\omega}_1) - i\,(\rho \sinh \omega_1 - \bar{\rho} \sinh \bar{\omega}_1) \\ - i\,(r \cos \sigma \sinh \omega_1 - \bar{r} \cos \bar{\sigma} \sinh \bar{\omega}_1)]\, X_2 \\ + (r \sin \sigma + \bar{r} \sin \bar{\sigma})\, X. \end{aligned}\right\} \quad (59)$$

Hence the surface S'' is real.

Among all the surfaces of Bonnet with a given spherical representation the origin itself may be counted. In this case we associate with it the trihedron, with vertex at the origin, rotating in such a way that its axes are parallel to the corresponding axes of the trihedron associated with a surface of Bonnet having the given spherical representation. Hence, if we put x, y, z, equal to zero in (59), these equations define all the real surfaces with a given spherical representation, when p, ρ, r, σ, ω are given all the sets of values which satisfy (18) and (55); now E and G in (56) are zero also.

Since

$$\frac{\partial x'}{\partial u} = \sqrt{E'}\, X_1, \quad \frac{\partial x'}{\partial v} = \sqrt{G'}\, X_2; \quad \frac{\partial \bar{x}'}{\partial u} = \sqrt{\bar{E}'}\, X_1, \quad \frac{\partial \bar{x}'}{\partial v} = \sqrt{\bar{G}'}\, X_2,$$

where the bar indicates the conjugate function, for the surface defined by (59) (with $x = y = z = 0$) we have

$$\sqrt{E''} = \sqrt{E'} + \sqrt{\bar{E}'}, \qquad \sqrt{G''} = \sqrt{G'} + \sqrt{\bar{G}'}. \quad (60)$$

We consider several particular cases.

§ 7. *Particular Surfaces of Bonnet.*

Let $\sigma = \frac{\pi}{2}$; from (34) it follows that $\sigma_2 = \frac{\pi}{2}$ also. Now equations (55) reduce to

$$\left.\begin{aligned} \cosh \omega_1 \frac{\partial p}{\partial u} - i \sinh \omega_1 \frac{\partial \rho}{\partial u} - (p \sinh \omega_1 - i\rho \cosh \omega_1) \sinh \omega \cosh \omega_1 = 0, \\ i \sinh \omega_1 \frac{\partial p}{\partial v} + \cosh \omega_1 \frac{\partial \rho}{\partial v} + (p \cosh \omega_1 - i\rho \sinh \omega_1) \cosh \omega \sinh \omega_1 = 0, \\ \frac{\partial r}{\partial u} + (ip \sinh \omega_1 + \rho \cosh \omega_1) \cosh \omega = 0, \\ \frac{\partial r}{\partial v} - (p \cosh \omega_1 - i\rho \sinh \omega_1) \sinh \omega = 0; \end{aligned}\right\} \quad (61)$$

and by means of these equations the expressions (56) are reducible to

$$\left.\begin{aligned}\sqrt{E} &= -\frac{1}{\cosh\omega_1}\frac{\partial\rho}{\partial u} + i\,p\sinh\omega + r\cosh\omega,\\ \sqrt{G} &= \frac{i}{\sinh\omega_1}\frac{\partial\rho}{\partial v} - p\cosh\omega + i\,r\sinh\omega\,;\end{aligned}\right\}\quad(62)$$

the accents have been removed.

Since equations (18) become

$$\frac{\partial\omega_1}{\partial u} + i\frac{\partial\omega}{\partial v} = -\sinh\omega\cosh\omega_1, \qquad i\frac{\partial\omega_1}{\partial v} + \frac{\partial\omega}{\partial u} = \cosh\omega\sinh\omega_1, \qquad (63)$$

three functions α, β, γ may be defined in the following way:

$$\left.\begin{aligned}d\alpha &= \sinh\omega\sinh\omega_1\,du + i\cosh\omega\cosh\omega_1\,dv,\\ d\beta &= -i\,e^{-\alpha}\left[\sinh\omega\cosh\omega_1\,du + i\sinh\omega_1\cosh\omega\,dv\right],\\ d\gamma &= -i\,e^{\alpha}\left[\cosh\omega\sinh\omega_1\,du + i\cosh\omega_1\sinh\omega\,dv\right].\end{aligned}\right\}\quad(64)$$

If we put

$$\rho = c, \qquad (65)$$

where c is a constant, the most general solution of equations (61) is

$$p = e^{\alpha}(\beta c + h), \qquad r = \gamma(c\beta + h) - c\tau, \qquad (66)$$

where h is an arbitrary constant and τ is given by

$$\left.\begin{aligned}\frac{\partial\tau}{\partial u} &= (-i\,e^{-\alpha}\gamma\sinh\omega + \cosh\omega)\cosh\omega_1,\\ \frac{\partial\tau}{\partial v} &= (e^{-\alpha}\gamma\cosh\omega + i\sinh\omega)\sinh\omega_1.\end{aligned}\right\}\quad(67)$$

We have neglected an additive constant for r, since it only tends to replace the surface now defined by surfaces parallel to it.

When c and h in (66) are real, all the surfaces of Bonnet defined by (59), with $x = 0$ and p, q, r given by (65) and (66), are evidently homothetic to the surfaces which are the loci of the points dividing in constant ratios the joins of corresponding points on the two surfaces for which

$$c = 0, \quad h = 1\,; \qquad c = 1, \quad h = 0.^{*}$$

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* cf. Surfaces Analogous to Surfaces of Bianchi, l. c. p. 121.

When ρ is not constant, the first two of equations (55) may be written

$$\left.\begin{aligned}\frac{\partial}{\partial u}\, e^{-\alpha}\, p &= i\, e^{-\alpha}\tanh\omega_1 \frac{\partial\rho}{\partial u} + \rho\,\frac{\partial\beta}{\partial u},\\ \frac{\partial}{\partial v}\, e^{-\alpha}\, p &= i\, e^{-\alpha}\coth\omega_1 \frac{\partial\rho}{\partial v} + \rho\,\frac{\partial\beta}{\partial v}.\end{aligned}\right\}\quad(68)$$

If p be eliminated from these equations by differentiating with respect to v and u respectively, it is found that ρ must satisfy the equation

$$\frac{\partial^2\rho}{\partial u\,\partial v} - \frac{\partial}{\partial v}\log\cosh\omega_1\frac{\partial\rho}{\partial u} - \frac{\partial}{\partial u}\log\sinh\omega_1\frac{\partial\rho}{\partial v} = 0.$$

But this equation is satisfied by the function expressing the distance from the origin to the plane tangent to any surface of Bonnet whose spherical representation is given by (21). Hence if we know a solution ω_1 of equations (63) and also a surface with the representation (21), we can find by quadratures a surface with the representation (2).

It is easy to furnish an illustration of this remark. Corresponding to equations (63) for the representation (2), we have for the representation (21)

$$\frac{\partial\omega_3}{\partial u} + i\,\frac{\partial\omega_1}{\partial v} = -\sinh\omega_1\cosh\omega_3,\qquad i\,\frac{\partial\omega_3}{\partial v} + \frac{\partial\omega_1}{\partial u} = \cosh\omega_1\sinh\omega_3.$$

A particular solution of these equations is

$$\omega_3 = \omega + i\,\pi.$$

Referring to (64), (65) and (66), we see that a surface with the representation (21) is defined by

$$x_1 = e^{-\alpha}\left(i\sinh\omega\, X_1' - \cosh\omega\, X_2'\right) - \beta\, X', \qquad (69)$$

and similar equations for y_1, z_1; X_1', X_2', X' being the direction-cosines of the tangents to the parametric curves on the representation (21) and of the radius to the point on the latter with respect to the fixed x-axis. Now the distance of the tangent plane from the origin is $-\beta$. If this be substituted in (68), we have for the functions p and ρ determining a surface with the representation (2),

$$p = \tfrac{1}{2}\left\{e^{-\alpha} - e^{\alpha}\left(\beta^2 + k\right)\right\},\qquad \rho = -\beta,$$

where k is an arbitrary constant, and r is given by quadratures from (55). This case follows from (66) by taking $h = 1$, $c = 0$ to determine (69); it is evident that other solutions can be found by quadratures, when these constants are given other values.

In a similar manner we can find a large number of surfaces of Bonnet by methods analogous to those which we have used in getting the surfaces analogous to surfaces of Bianchi.*

§ 8. *Determination of the Surface of Bonnet Applicable to a Given Surface of Bonnet.*

We have seen in § 5 that if σ' is defined by (44) the function ω_1 gives a Bäcklund transformation of a surface with the spherical representation (1). We will now use this fact to obtain a general method of determining surfaces with this representation similar to that established in § 6. Instead of starting with a surface having this representation we take the origin and associate with it a trihedron whose axes are parallel to the axes of the trihedron associated with the sperical surface having this representation.

If we denote by p', ρ', r' the functions analogous to p, q, r, as defined in § 6, the coordinates with respect to the fundamental trihedron of a point on a surface of the group can be written, in consequence of (44),

$$-[p' \cosh \omega_1 - i(\rho' + r' \sec \sigma) \sinh \omega_1], \quad -[ip' \sinh \omega_1 + (\rho' + r' \sec \sigma) \cosh \omega_1],$$
$$i r' \tan \sigma.$$

Expressions for the projections upon the axes of a displacement upon the surface are similar in form to (54); from these it is found that the necessary and sufficient condition that the surface be a surface of Bonnet, with the given spherical representation of its lines of curvature, is that p', q', r' satisfy the conditions

$$\left.\begin{aligned} &\left(i \sinh \omega_1 \frac{\partial p'}{\partial u} + \cosh \omega_1 \frac{\partial \rho'}{\partial u}\right) \tan \sigma + i \sinh \omega_1 \cosh \omega \\ &\qquad\qquad [p' \cosh \omega_1 - i(\rho' + r' \sec \sigma) \sinh \omega_1] = 0, \\ &\left(i \cosh \omega_1 \frac{\partial p'}{\partial v} + \sinh \omega_1 \frac{\partial \rho'}{\partial v}\right) \tan \sigma - \cosh \omega_1 \sinh \omega \\ &\qquad\qquad [p' \sinh \omega_1 - i(\rho' + r \sec \sigma) \cosh \omega_1] = 0, \\ &\tan \sigma \frac{\partial r'}{\partial u} = + [i p' \cosh \omega_1 + (\rho' + r' \sec \sigma) \sinh \omega_1] \sinh \omega, \\ &\tan \sigma \frac{\partial r'}{\partial v} = - [p' \sinh \omega_1 - (\rho' + r' \sec \sigma) i \cosh \omega_1] \cosh \omega. \end{aligned}\right\} \quad (70)$$

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* l. c. pp. 125–134.

The coefficients of the linear element of the surface are given by

$$\left.\begin{aligned}\sqrt{E_1} &= -\cosh\omega_1\frac{\partial p'}{\partial u} + i\sinh\omega_1\frac{\partial\rho'}{\partial u} - i(\rho'\sec\sigma + r')\sinh\omega\cot\sigma \\ &\quad - \sinh\omega_1\cosh\omega\cot\sigma\,[p'\sinh\omega_1 - (\rho' + r'\sec\sigma)\,i\cosh\omega_1], \\ \sqrt{G_1} &= -i\sinh\omega_1\frac{\partial p'}{\partial v} - \cosh\omega_1\frac{\partial\rho'}{\partial v} - i(\rho'\sec\sigma + r')\cosh\omega\cot\sigma \\ &\quad + \cosh\omega_1\sinh\omega\cot\sigma\,[p'\cosh\omega_1 - i(\rho' + r'\sec\sigma)\sinh\omega_1].\end{aligned}\right\}\quad(71)$$

Suppose now that we have given a surface, S, of Bonnet with the representation (2) and a solution ω_1 of equations (18). For S the functions p, ρ, r are known. From the equations

$$\sqrt{E_1} = \sqrt{E}, \qquad \sqrt{G_1} = \sqrt{G},$$

when substitution has been made from (56)* and (71), and the first two of equations (70) we get $\frac{\partial p'}{\partial u}$, $\frac{\partial p'}{\partial v}$, $\frac{\partial\rho'}{\partial u}$, $\frac{\partial\rho'}{\partial v}$ in terms of known quantities. The conditions of integrability of these expressions and the last two of (70) are reducible to three linear equations in p', ρ', r'. Thus we find by algebraic processes p', q', r', determining the unique surface S_1 applicable to a given surface S with correspondence of lines of curvature; it has been shown that there always is a surface S_1 of the kind sought.

If σ is such that condition (52) is satisfied, we can get at once another pair of applicable real surfaces of Bonnet as shown in § 5.

PRINCETON, *January*, 1906.

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* Here $\sqrt{E}$ is $\sqrt{E'}$ of (56) and E of the latter is zero.

On the Factoring of Composite Hypercomplex Number Systems.[*]

By Heman Burr Leonard.

Introduction.

From the two number systems $E \equiv e_1 \ldots e_n$ and $F \equiv f_1 \ldots f_r$, having the multiplication tables $e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3}$ $(i = 1, \ldots, n)$ and $f_{j_1} f_{j_2} = \sum_{j_3} \phi_{j_1 j_2 j_3} f_{j_3}$ $(j = 1, \ldots, r)$, can be formed by multiplication[†] a number system of nr units $\varepsilon_{i_1 j_1} = e_{i_1} f_{j_1} = f_{j_1} e_{i_1}$, having the multiplication table $\varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2} = (e_{i_1} e_{i_2})\ (f_{j_1} f_{j_2}) = \sum_{i_3 j_3} \gamma_{i_1 i_2 i_3} \phi_{j_1 j_2 j_3} \varepsilon_{i_3 j_3}$. In regard to the converse problem Professor Scheffers suggested[‡] in 1891 that there was lacking a serviceable criterion for deciding whether a given system is a compound of systems and also that general theorems concerning the divisors of zero and the characteristic equation were desirable The consideration of these questions has led to the results which are now given in what is to be regarded as a first communication.

Let $A = \sum_i a_i e_i$ and $\bar{A} = \sum_j \bar{a}_j f_j$ be numbers of the systems E and F respectively. Then the number $C = \sum_{ij} a_i \bar{a}_j \varepsilon_{ij}$ *will be called the compound of the numbers* A *and* $\bar{A}$. It is shown in §2 that *if* $\mu_1, \ldots, \mu_n$ *are the roots of the characteristic equation of* A, *and* $\nu_1, \ldots, \nu_r$ *are the roots of the characteristic equation of* $\bar{A}$, *then the roots of the characteristic equation of* C *are* $\mu_i \nu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, r)$.

In §3 is given a method for determining the factor systems of a composite system through the use of the characteristic equation of the composite system

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[*] This paper was read at the meeting of the American Mathematical Society, held at Yale University, September, 1906. An abstract appears in the Bulletin, vol. 13, number 2 (November, 1906), p. 68.

[†] Scheffers, Mathematische Annalen, vol. 39 (1891), p. 324.

[‡] Annalen, vol. 39 (1891), p. 325. "Es fehlt ein brauchbares Criterium dafür, dass ein vorgelegtes System als Product aufgefasst werden kann, und an allgemeinen Sätzen über die Theiler der Null und die charakteristische Gleichung eines solchen Systems."

The method is made clear by its application to the factoring of two composite systems.

A second method, which uses the matrix representation, is given in §4. Because of the difficulty of solving algebraic equations of higher degree than the fourth, this method appears to be the more serviceable one for decomposing composite algebras of the higher orders.

In §5 divisors of zero are considered.

§1.—THE GROUP OF THE COMPOUND SYSTEM.

According to Poincaré[*] and Study[†] the groups of the algebras E and F are respectively

$$\left.\begin{aligned} G_E : x'_{i_3} &= \sum_{i_1 i_2} \gamma_{i_1 i_2 i_3}\, y_{i_2}\, x_{i_1},\ (i = 1, \ldots, n); \\ G_F : \bar{x}'_{j_3} &= \sum_{j_1 j_2} \phi_{j_1 j_2 j_3}\, \bar{y}_{j_2}\, \bar{x}_{j_1},\ (j = 1, \ldots, r); \end{aligned}\right\} \tag{1}$$

where the x's are variables, the y's parameters. If $X = \sum_{i_1 j_1} x_{i_1 j_1}\, e_{i_1}\, f_{j_1}$, $Y = \sum_{i_2 j_2} y_{i_2 j_2}\, e_{i_2}\, f_{j_2}$, $Z = \sum_{i_3 j_3} z_{i_3 j_3}\, e_{i_3}\, f_{j_3}$, are numbers of the compound algebra $EF \equiv e_i f_j\, (i = 1, \ldots, n\,;\, j = 1, \ldots, r)$, such that $Z = XY$, then the group of the compound algebra is

$$G_{EF} : z_{i_3 j_3} = \sum_{i_1 i_2 j_1 j_2} \gamma_{i_1 i_2 i_3}\, \phi_{j_1 j_2 j_3}\, y_{i_2 j_2}\, x_{i_1 j_1}\ (i = 1, \ldots, n\,;\ j = 1, \ldots, r). \tag{2}$$

According to Rados[‡] and Burnside,[§] the compound $G_E G_F$ of the groups G_E, G_F is obtained as follows: In the function $f = \sum_{i_3 j_3} c'_{i_3 j_3}\, x'_{i_3}\, \bar{x}'_{j_3}$ substitute the values of x'_{i_3} and $\bar{x}'_{j_3}$ and equate the resulting form to $\sum_{i_1 j_1} c_{i_1 j_1}\, x_{i_1}\, \bar{x}_{j_1}$. By comparing coefficients there results

$$c_{i_1 j_1} = \sum_{i_3 i_2 j_3 j_2} \gamma_{i_1 i_2 i_3}\, \phi_{j_1 j_2 j_3}\, y_{i_2}\, \bar{y}_{j_2}\, c'_{i_3 j_3}.\|$$

Therefore the compound of the groups G_E, G_F may be written

$$G_E G_F : c_{i_1 j_1} = \sum_{i_3 i_2 j_3 j_2} \gamma_{i_1 i_2 i_3}\, \phi_{j_1 j_2 j_3}\, y_{i_2 j_2}\, c'_{i_3 j_3}. \tag{3}$$

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[*] Poincaré, Comptes Rendus, vol. 99 (1884), pp. 740-742.

[†] Study, Monatshefte für Math. und Physik, vol. 1 (1890), pp. 283-355.

[‡] Rados, Annalen, vol. 48 (1897), pp. 417-424.

[§] Burnside, Quarterly Journal of Mathematics, vol. 33 (1902), pp. 80-84.

[‖] According to a suggestion derived from a paper by Franklin, this may be called the induced group of G_E and G_F. American Journal of Mathematics, vol. 16 (1894), p. 205.

It can be easily seen that the *transverse*[*] or *converse* or *conjugate* of the group $G_E\,G_F$, designated by $\overset{\smile}{G_E G_F}$, is a subgroup of G_{EF}.

§2.—THE ROOTS OF THE CHARACTERISTIC EQUATION OF THE COMPOUND SYSTEM.

The characteristic equation of E is obtained by writing in the equations of the group G_E $x'_{i_3} = \mu\, y_{i_3}$,[†] transposing, and since at least one y_i does not vanish, equating the determinant of the coefficients to zero:

$$\begin{vmatrix} \sum\limits_{i_1} x_{i_1}\gamma_{i_1 11} - \mu, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 21}, & \ldots, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 n1} \\ \sum\limits_{i_1} x_{i_1}\gamma_{i_1 12}, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 22} - \mu, & \ldots, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 n2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{i_1} x_{i_1}\gamma_{i_1 1n}, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 2n}, & \ldots, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 nn} - \mu \end{vmatrix} = 0. \qquad (4)$$

From another point of view the scalar μ must satisfy the equation (4) in order that for the general number x of E there should exist a number y such that $xy = \mu y$.[‡]

Similarly, if one writes $\nu\bar{y}_{j_3}$ for $\bar{x}'_{j_3}$ in the equations of the group G_F and transposes, the determinant of the coefficients of the y's

$$\begin{vmatrix} \sum\limits_{j_1} \bar{x}_{j_1}\phi_{j_1 j_2 j_3} - \nu\delta_{j_2 j_3}\,\S \\ j_2,\ j_3 = 1, \ldots, r \end{vmatrix} = 0, \qquad (5)$$

expresses the fact that there exists a $y \neq 0$, such that $\bar{x}\bar{y} = \nu\bar{y}$.

The characteristic equation of the compound system EF[||] obtained in the same manner from the group G_{EF} is:

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[*] In the American Journal of Mathematics, vol. 12 (1890), p. 340, Taber attributes the term transverse to Cayley, the term converse to Charles Peirce, and the term conjugate to Hamilton.

[†] Scheffers, Annalen, vol. 39 (1891), p. 303.

[‡] If we let $x'_{i_3} = \mu x_{i_3}$, an equation similar to (4) is obtained, which expresses the fact that a number y exists such that $yx = \mu y$. In the present investigation, we follow Cartan (Annales de la Faculté des Sciences de Toulouse, vol. 12 (1898), p. B 17) in restricting our attention to the equation (4).

[§] Here and hereafter in this paper $\delta_{j_2 j_3} = \begin{cases} 1, \text{ for } j_2 = j_3 \\ 0, \text{ for } j_2 \neq j_3 \end{cases}$ according to the Kronecker usage.

[||] At first glance one might surmise that the characteristic equation of the compound system EF should be

$$\begin{vmatrix} \sum\limits_{i_1} x_{i_1}\gamma_{i_1 i_2 i_3} - \mu\delta_{i_2 i_3} \\ i_2,\ i_3 = 1, \ldots, n \end{vmatrix} \cdot \begin{vmatrix} \sum\limits_{j_1} \bar{x}_{j_1}\phi_{j_1 j_2 j_3} - \mu\delta_{j_2 j_3} \\ j_2,\ j_3 = 1, \ldots, r \end{vmatrix} = 0,$$

which is in fact the characteristic equation of the reducible system having E and F for its constituents. Annalen, vol. 39 (1891), p. 320.

$$\left|\begin{array}{ccccccccc}
\sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 11}-\zeta, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 21} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 r1} & , & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 21}\,\Phi_{j_1 11} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 n1}\,\Phi_{j_1 r1} \\
\sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 12}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 22}-\zeta & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 r2} & , & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 21}\,\Phi_{j_1 12} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 n1}\,\Phi_{j_1 r2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 1r}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 2r} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 11}\,\Phi_{j_1 rr}-\zeta & , & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 21}\,\Phi_{j_1 1r} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 n1}\,\Phi_{j_1 rr} \\
\sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 12}\,\Phi_{j_1 11}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 12}\,\Phi_{j_1 21} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 12}\,\Phi_{j_1 r1} & , & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 22}\,\Phi_{j_1 11}-\zeta & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 n2}\,\Phi_{j_1 r1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 1n}\,\Phi_{j_1 1r}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 1n}\,\Phi_{j_1 2r} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 1n}\,\Phi_{j_1 rr} & , & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 2n}\,\Phi_{j_1 1r} & ,\cdots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\,\gamma_{i_1 nn}\,\Phi_{j_1 rr}-\zeta
\end{array}\right| = 0. \qquad (6)$$

The characteristic equation of the special number $A = a_i e_i$ of E is obtained from (4) by writing $a_{i_1} = x_{i_1}$. Similarly the characteristic equation of $\bar{A} = \bar{a}_j f_j$ is obtained from (5) by writing $\bar{a}_{j_1} = \bar{x}_{j_1}$. Since the characteristic equation of a matrix is the same as that of its conjugate, the characteristic equation of the compound C of these numbers (Introduction) is obtained by writing $a_{i_1}\bar{a}_{j_1} = x_{i_1 j_1}$ in (6). We proceed to show that if the roots of

$$\left|\begin{matrix}\sum\limits_{i_1} a_{i_1}\gamma_{i_1 i_2 i_3} - \mu\delta_{i_2 i_3} \\ i_2,\, i_3 = 1, \ldots, n\end{matrix}\right| = 0 \qquad (4')$$

are $\mu_1, \ldots, \mu_n$ and those of

$$\left|\begin{matrix}\sum\limits_{j_1} \bar{a}_{j_1}\phi_{j_1 j_2 j_3} - \nu\delta_{j_2 j_3} \\ j_2,\, j_3 = 1, \ldots, r\end{matrix}\right| = 0 \qquad (5')$$

are $\nu_1, \ldots, \nu_r$, then the nr roots of the characteristic equation of the compound number C

$$\left|\begin{matrix}\sum\limits_{i_1 j_1} a_{i_1}\bar{a}_{j_1}\gamma_{i_1 i_2 i_3}\phi_{j_1 j_2 j_3} - \zeta\delta_{i_2 i_3}\delta_{j_2 j_3} \\ j_2 = 1, \ldots, r;\ \text{then}\ i_2 = 1, \ldots, n \\ j_3 = 1, \ldots, r;\ \text{then}\ i_3 = 1, \ldots, n\end{matrix}\right| = 0 \qquad (6')$$

are $\mu_i\nu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, r)$.

If $\mu_1, \ldots, \mu_n$ are the roots of the equation (4′), there are n linear functions $L_1, \ldots, L_n$, which are transformed by any particular substitution S_E of the group G_E into $\mu_1 L_1, \ldots, \mu_n L_n$. Likewise if $\nu_1, \ldots, \nu_r$ are the roots of (5′), there exist r linear functions $\bar{L}_1, \ldots, \bar{L}_r$, which are transformed by any particular substitution S_F of the group G_F into $\nu_1\bar{L}_1, \ldots, \nu_r\bar{L}_r$. Evidently the functions $L_i\bar{L}_j$ are transformed by the successive operation S_E, S_F into $\mu_i\nu_j L_i\bar{L}_j$ The same result is obtained by transforming $L_i\bar{L}_j$ by $\underbrace{S_E S_F}$. But $\underbrace{S_E S_F}$ $(L_i\bar{L}_j)$ $= \zeta_{ij} L_i\bar{L}_j$.* Therefore $\zeta_{ij} = \mu_i\nu_j$ and the theorem is proved.

§3.—FACTORING OF COMPOSITE SYSTEMS BY CHARACTERISTIC EQUATION METHOD.

I. The multiplication tables of the systems E and F being given, the multiplication table of the compound system EF is determined by the consideration that its nr units are $e_i f_j$. If the characteristic equations of a number A of E and a number $\bar{A}$ of F are given, the characteristic equation of the compound number C can be determined. Let

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* Franklin, American Journal of Mathematics, vol. 16 (1894), p. 205.

$$\mu^n - p_1\mu^{n-1} + p_2\mu^{n-2} - \dots + (-1)^n p_n = 0 \tag{4''}$$

and

$$\nu^r - q_1\nu^{r-1} + q_2\nu^{r-2} - \dots + (-1)^r q_r = 0 \tag{5''}$$

be the characteristic equations of A and $\bar{A}$ and let

$$\zeta^{nr} - s_1\zeta^{nr-1} + s_2\zeta^{nr-2} - \dots + (-1)^{nr} s_{nr} = 0 \tag{6''}$$

be the characteristic equation of the compound number C. The coefficients s can be determined in terms of p and q. Since the roots of (6″) are $\mu_i\nu_j$, the coefficients s of (6″) are calculated in terms of p and q by means of the symmetric functions of the roots of (4″) and (5″). The converse problem is considered from two points of view. In §4 from a given compound system are derived the factor systems. In this section (§3) from the characteristic equation of a general number C of the compound system are calculated the characteristic equations of corresponding general numbers A and $\bar{A}$ of the factor systems.

That the problems of §4 and §3 are not strictly identical can best be made clear by an illustration. The characteristic equation of a general number of the system

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_1	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_2	h_2	0	h_4	0	h_6	0	h_8	0
h_3	h_3	h_4	h_7	h_8	0	0	0	0
h_4	h_4	0	h_8	0	0	0	0	0
h_5	h_5	h_6	0	0	0	0	0	0
h_6	h_6	0	0	0	0	0	0	0
h_7	h_7	h_8	0	0	0	0	0	0
h_8	h_8	0	0	0	0	0	0	0

is $(x_1 - \zeta)^8 = 0$. By the methods explained later in this section, the characteristic equations of general numbers of the factor systems are calculated to be $\mu^2 - p_1\mu + \frac{p_1^2}{4} = 0$ and $\nu^4 - q_1\nu^3 + \frac{3}{8}q_1^2\nu^2 - \frac{1}{16}q_1^3\nu + \frac{1}{256}q_1^4 = 0$. The first one is evidently the characteristic equation of a general number of the Cayley two-unit system.* On the other hand the system belonging to $\left(v - \frac{q_1}{4}\right)^4 = 0$ is not uniquely determined, since all of the following systems have the same equation:†

IV_1.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_3	e_4	0
e_3	e_3	e_4	0	0
e_4	e_4	0	0	0

IV_3.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	λe_4	e_4	0
e_3	e_3	$-e_4$	e_4	0
e_4	e_4	0	0	0

IV_4.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	0	0
e_3	e_3	0	e_4	0
e_4	e_4	0	0	0

IV_5.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

IV_8.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	0	e_4	0
e_3	e_3	$-e_4$	0	0
e_4	e_4	0	0	0

IV_9.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	0	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

However by the method of §4 the factor systems are found to be

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

and

	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_4	0	0
f_3	f_3	0	0	0
f_4	f_4	0	0	0.

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* $e^2_1 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e^2_2 = 0$.

† Scheffers, Annalen, vol. 39 (1891), p. 352. The characteristic equations there given are in the reduced form.

Nevertheless in some respects the method of this section is more powerful than that of §4. Thus the system

$$\begin{array}{c|cccc} & h_1 & h_2 & h_3 & h_4 \\ \hline h_1 & h_1 & h_2 & h_3 & h_4 \\ h_2 & h_2 & 0 & h_4 - h_3 & h_4 - h_3 \\ h_3 & h_3 & h_4 - h_3 & -h_1 & -h_1 - h_2 \\ h_4 & h_4 & h_4 - h_3 & -h_1 - h_2 & -h_1 - 2h_2 \end{array}$$

can not be resolved by the method of §4; but the characteristic equation of its general number is $\zeta^4 - 4x_1\zeta^3 + \zeta^2(6x_1^2 + 2x_3^2 + 4x_3x_4 + 2x_4^2) - \zeta(4x_1^3 + 4x_1x_3^2 + 8x_1x_3x_4 + 4x_1x_4^2) + (x_1^4 + 2x_1^2x_3^2 + 4x_1^2x_3x_4 + 2x_1^2x_4^2 + x_3^4 + 4x_3^3x_4 + 6x_3^2x_4^2 + 4x_3x_4^3 + x_4^4) = 0$ and by the method of this section the characteristic equations of general numbers of its factor systems are found to be $\mu^2 - p_1\mu + \frac{p_1^2}{4} = 0$ and $\nu^2 - q_1\nu + \frac{q_1^2}{4x_1^2}(x_1^2 + x_3^2 + x_4^2 + 2x_3x_4) = 0$. The factor systems* belong to the types

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & 0 \end{array} \quad \text{and} \quad \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & f_1 & f_2 \\ f_2 & f_2 & -f_1. \end{array}$$

II. We start with the simplest composite systems, namely those of order four, whose factors must be two two-unit systems. Assume as the characteristic equation of a general number of the compound system $\zeta^4 - s_1\zeta^3 + s_2\zeta^2 - s_3\zeta + s_4 = 0$. For the characteristic equations of general numbers of the two factor systems may be assumed $\mu^2 - p_1\mu + p_2 = 0$ and $\nu^2 - q_1\nu + q_2 = 0$. By forming the symmetric functions of the roots of these equations, the following relations are obtained:

$$\left.\begin{aligned} s_1 &= p_1 q_1 \\ s_2 &= p_1^2 q_2 - 2p_2 q_2 + p_2 q_1^2 \\ s_3 &= p_1 p_2 q_1 q_2 \\ s_4 &= p_2^2 q_2^2. \end{aligned}\right\} \qquad (7)$$

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* When these two systems are compounded and the following linear transformations are made on the units, $h_4 = g_3 + g_4$, $h_1 = g_1$, $h_2 = g_2$, $h_3 = g_3$, the given form of the composite system results.

An obvious condition on the s's is $s_1^2 s_4 = s_3^2$. The formation of the characteristic equation for a general number of the given system furnishes the values for the s's. From the above relations p_2 can be determined in terms of p_1 and q_2 in terms of q_1. Thus the nature of the roots of the characteristic equations of general numbers of the two factor systems is determined.

For example, consider the system

	h_1	h_2	h_3	h_4
h_1	h_1	h_2	h_3	h_4
h_2	h_2	0	h_4	0
h_3	h_3	h_4	$-h_1$	$-h_2$
h_4	h_4	0	$-h_2$	0.

The characteristic equation of a general number of the system is

$$\begin{vmatrix} x_1-\zeta, & 0\;, & -x_3, & 0 \\ x_2\;, & x_1-\zeta, & -x_4, & -x_3 \\ x_3\;, & 0\;, & x_1-\zeta, & 0 \\ x_4\;, & x_3\;, & x_2\;, & x_1-\zeta \end{vmatrix} = 0,$$

which, multiplied out, is

$$\zeta^4-\zeta^3(4x_1)+\zeta^2(6x_1^2+2x_3^2)-\zeta(4x_1^3+4x_1x_3^2)+(x_1^4+2x_1^2x_3^2+x_3^4)=0.$$

Substituting in the above relations (7)

$$\begin{aligned} p_1q_1 &= 4x_1 \\ p_1^2q_2-2p_2q_2+p_2q_1^2 &= 6x_1^2+2x_3^2 \\ p_1p_2q_1q_2 &= 4x_1(x_1^2+x_3^2) \\ p_2^2q_2^2 &= x_1^4+2x_1^2x_3^2+x_3^4=(x_1^2+x_3^2)^2. \end{aligned}$$

Combining and solving, the following values for the coefficients are obtained:

$$p_2=\frac{p_1^2}{4} \quad\text{or}\quad \frac{p_1^2(x_1^2+x_3^2)}{4x_1^2}$$

$$q_2=\frac{q_1^2(x_1^2+x_3^2)}{4x_1^2} \quad\text{or}\quad \frac{q_1^2}{4}.$$

Substituting the first set of these values, the characteristic equations of general numbers of the factor systems become $\mu^2 - p_1\mu + \frac{p_1^2}{4} = 0$ and $\nu^2 - q_1\nu + \frac{q_1^2(x_1^2 + x_3^2)}{4x_1^2} = 0$. The first has equal roots and indicates the Cayley system. The second has complex roots and indicates the ordinary complex system.

The substitution of the second set of these values gives the same equations in reverse order.

III. The second lowest composite number is six and a compound system of six units must have for its factor systems a two-unit and a three-unit system. Assume as the characteristic equation of a general number of the composite system

$$\zeta^6 - s_1\zeta^5 + s_2\zeta^4 - s_3\zeta^3 + s_4\zeta^2 - s_5\zeta + s_6 = 0$$

and for general numbers of the two factor systems $\mu^3 - p_1\mu^2 + p_2\mu - p_3 = 0$ and $\nu^2 - q_1\nu + q_2 = 0$. By forming the symmetric functions of the roots of these equations, the following relations are obtained:

$$\left.\begin{aligned} s_1 &= p_1 q_1 \\ s_2 &= p_1^2 q_2 - 2p_2 q_2 + p_2 q_1^2 \\ s_3 &= p_1 p_2 q_1 q_2 + p_3 q_1^3 - 3p_3 q_1 q_2 \\ s_4 &= p_2^2 q_2^2 + p_1 p_3 q_1^2 q_2 - 2p_1 p_3 q_2^2 \\ s_5 &= p_2 p_3 q_1 q_2^2 \\ s_6 &= p_3^2 q_2^3. \end{aligned}\right\} \qquad (8)$$

The formation of the characteristic equation for a general number of the given system furnishes the values for the *s's*. From the above relations p_2 and p_3 can be determined in terms of p_1, and q_2 in terms of q_1. This enables one to decide the nature of the roots of the characteristic equations of general numbers of the two factor systems.*

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* The above six equations contain five unknowns p_1, p_2, p_3, q_1, q_2, the elimination of which gives certain syzygies among the *s's*. When these relations are fulfilled, the number (whose characteristic equation is being considered) is a compound. The eliminations of the *p's* and *q's* are too lengthy to be taken up at present.

For example, consider the system

$$\begin{array}{c|cccccc}
 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\
\hline
h_1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\
h_2 & h_2 & 0 & h_4 & 0 & h_6 & 0 \\
h_3 & h_3 & h_4 & 0 & 0 & 0 & 0 \\
h_4 & h_4 & 0 & 0 & 0 & 0 & 0 \\
h_5 & h_5 & h_6 & 0 & 0 & 0 & 0 \\
h_6 & h_6 & 0 & 0 & 0 & 0 & 0.
\end{array}$$

The characteristic equation of a general number of this system is

$$\zeta^6 - 6x_1\zeta^5 + 15x_1^2\zeta^4 - 20x_1^3\zeta^3 + 15x_1^4\zeta^2 - 6x_1^5\zeta + x_1^6 = 0.$$

Substituting in the above relations (8)

$$\begin{aligned}
p_1 q_1 &= 6x_1 \\
p_1^2 q_2 - 2p_2 q_2 + p_2 q_1^2 &= 15x_1^2 \\
p_1 p_2 q_1 q_2 + p_3 q_1^3 - 3p_3 q_1 q_2 &= 20x_1^3 \\
p_2^2 q_2^2 + p_1 p_3 q_1^2 q_2 - 2p_1 p_3 q_2^2 &= 15x_1^4 \\
p_2 p_3 q_1 q_2^2 &= 6x_1^5 \\
p_3^2 q_2^3 &= x_1^6.
\end{aligned}$$

Combining and solving, from the first, second, fifth, and sixth of these relations the following values for the coefficients are obtained:

$$p_1 = -4p_3^{\frac{1}{3}},\ 3p_3^{\frac{1}{3}},\ \text{or}\ 3p_3^{\frac{1}{3}}.$$

With the first of these values are associated

$$\begin{aligned}
p_2 &= -\tfrac{1}{4}p_1^2 \\
p_3 &= -\tfrac{1}{64}p_1^3 \\
q_1 &= \frac{6x_1}{p_1} \\
q_2 &= \frac{16x_1^2}{p_1^2} = \frac{4q_1^2}{9}.
\end{aligned}$$

With the second of these values are associated

$$\begin{aligned}
p_2 &= \tfrac{1}{3}p_1^2 \\
p_3 &= \tfrac{1}{27}p_1^3 \\
q_1 &= \frac{6x_1}{p_1} \\
q_2 &= \frac{9x_1^2}{p_1^2} = \tfrac{1}{4}q_1^2.
\end{aligned}$$

In calculating these values the third relation and the fourth relation were not used and we find that the third relation is not satisfied by the first set of values. The other set of values satisfies all six relations. The characteristic equations of general numbers of the factor systems are therefore

$$\nu^2 - q_1 \nu + \tfrac{1}{4} q_1^2 = 0, \qquad \nu = \tfrac{1}{2} q_1, \qquad \tfrac{1}{2} q_1,$$

and

$$\mu^3 - p_1 \mu^2 + \tfrac{1}{3} p_1^2 \mu - \tfrac{1}{27} p_1^3 = 0, \qquad \mu = \tfrac{1}{3} p_1, \qquad \tfrac{1}{3} p_1, \qquad \tfrac{1}{3} p_1.$$

The first has equal roots and indicates the Cayley two-unit system. The second has three equal roots and indicates one of the two types of systems *

$$\text{III}_1. \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_2 & e_3 \\ e_2 & e_2 & e_3 & 0 \\ e_3 & e_3 & 0 & 0 \end{array} \qquad \text{and} \qquad \text{III}_3. \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_2 & e_3 \\ e_2 & e_2 & 0 & 0 \\ e_3 & e_3 & 0 & 0. \end{array}$$

Trial shows that the second system is the desired three-unit factor system.

IV. The treatment of composite systems of higher orders has been made along similar lines, but it is not considered advisable to take space at this time to give the details.

§4.—FACTORING OF COMPOSITE SYSTEMS BY MATRIX METHOD.

I. "The relative form" of an associative algebra, developed by Charles S. Peirce,† is really a representation of the algebra in matrix form.‡ Given the multiplication table of an algebra in the form

$$\left.\begin{array}{l} e_1^2 \;\, = \gamma_{111} e_1 + \gamma_{112} e_2 + \gamma_{113} e_3 + \cdots + \gamma_{11n} e_n \\ e_1 e_2 = \gamma_{121} e_1 + \gamma_{122} e_2 + \gamma_{123} e_3 + \cdots + \gamma_{12n} e_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_1 e_n = \gamma_{1n1} e_1 + \gamma_{1n2} e_2 + \gamma_{1n3} e_3 + \cdots + \gamma_{1nn} e_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_n^2 \;\, = \gamma_{nn1} e_1 + \gamma_{nn2} e_2 + \gamma_{nn3} e_3 + \cdots + \gamma_{nnn} e_n, \end{array}\right\} \tag{9}$$

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* Scheffers, Annalen, vol. 39 (1891), p. 353.

† Peirce, American Journal of Mathematics, vol. 4 (1882), p. 221. Proceedings of the American Academy of Arts and Sciences, May 11, 1875, whole series vol. 10, p. 392. Also Benjamin Peirce, same vol., p. 397.

‡ Shaw, Transactions of the American Mathematical Society, vol. 4 (1903), p. 252.

the matrix representation (the ideal units $I:A$, $J:A$, etc., excluded) is in double suffix notation[*]

$$\left.\begin{aligned}
g_1' = {}& \gamma_{111}g_{11} + \gamma_{112}g_{21} + \gamma_{113}g_{31} + \cdots\cdots + \gamma_{11n}g_{n1} \\
&+ \gamma_{121}g_{12} + \gamma_{122}g_{22} + \gamma_{123}g_{32} + \cdots\cdots + \gamma_{12n}g_{n2} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \gamma_{1n1}g_{1n} + \gamma_{1n2}g_{2n} + \gamma_{1n3}g_{3n} + \cdots\cdots + \gamma_{1nn}g_{nn} \\
g_2' = {}& \gamma_{211}g_{11} + \gamma_{212}g_{21} + \gamma_{213}g_{31} + \cdots\cdots + \gamma_{21n}g_{n1} \\
&+ \gamma_{221}g_{12} + \gamma_{222}g_{22} + \gamma_{223}g_{32} + \cdots\cdots + \gamma_{22n}g_{n2} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \gamma_{2n1}g_{1n} + \gamma_{2n2}g_{2n} + \gamma_{2n3}g_{3n} + \cdots\cdots + \gamma_{2nn}g_{nn} \\
g_i' = {}& \gamma_{i11}g_{11} + \gamma_{i12}g_{21} + \gamma_{i13}g_{31} + \cdots\cdots + \gamma_{i1n}g_{n1} \\
&+ \gamma_{i21}g_{12} + \gamma_{i22}g_{22} + \gamma_{i23}g_{32} + \cdots\cdots + \gamma_{i2n}g_{n2} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \gamma_{in1}g_{1n} + \gamma_{in2}g_{2n} + \gamma_{in3}g_{3n} + \cdots\cdots + \gamma_{inn}g_{nn}.
\end{aligned}\right\} \quad (10)$$

Taber proves that matrices of composite order can be factored.[†] From this a suggestion comes for factoring a composite algebra. Put the composite algebra into matrix form, factor the matrices, and translate the factors back into number systems.

On account of the difficulty of describing this method in words, it is placed before the reader in the solution of four examples. These illustrations are sufficient to make evident the scheme, which is perfectly general.

II. The first system to be considered is the one whose multiplication table is

	h_1	h_2	h_3	h_4
h_1	h_1	h_2	h_3	h_4
h_2	h_2	0	h_4	0
h_3	h_3	h_4	$-h_1$	$-h_2$
h_4	h_4	0	$-h_2$	0.

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[*] Study, Encyklopaedie der Math. Wissen., vol. 1, p. 170.

[†] Taber, American Journal of Mathematics, vol. 12 (1890), p. 391.

If this is a composite system, its factors must be two two-unit systems. Assume for them

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_{11} & e_{12} \\ e_2 & e_{21} & e_{22} \end{array} \quad \text{and} \quad \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22}. \end{array}$$

Symbolically the compound system is

$$\begin{array}{c|cccc} & e_1f_1 & e_2f_1 & e_1f_2 & e_2f_2 \\ \hline e_1f_1 & e_{11}f_{11}=g_{11} & e_{12}f_{11}=g_{12} & e_{11}f_{12}=g_{13} & e_{12}f_{12}=g_{14} \\ e_2f_1 & e_{21}f_{11}=g_{21} & e_{22}f_{11}=g_{22} & e_{21}f_{12}=g_{23} & e_{22}f_{12}=g_{24} \\ e_1f_2 & e_{11}f_{21}=g_{31} & e_{12}f_{21}=g_{32} & e_{11}f_{22}=g_{33} & e_{12}f_{22}=g_{34} \\ e_2f_2 & e_{21}f_{21}=g_{41} & e_{22}f_{21}=g_{42} & e_{21}f_{22}=g_{43} & e_{22}f_{22}=g_{44}. \end{array} \tag{11}$$

Substituting in the above formulas (10), the following expressions result

$$\begin{array}{rrrrr} g'_1 = & (\ \ 1)\,g_{11} + & 0\ g_{21} + & 0\ g_{31} + & 0\ g_{41} \\ + & 0\ g_{12} + & (\ \ 1)\,g_{22} + & 0\ g_{32} + & 0\ g_{42} \\ + & 0\ g_{13} + & 0\ g_{23} + & (1)\,g_{33} + & 0\ g_{43} \\ + & 0\ g_{14} + & 0\ g_{24} + & 0\ g_{34} + & (1)\,g_{44} \\ = & g_{11} + & g_{22} + & g_{33} + & g_{44}, \end{array}$$

$$\begin{array}{rrrrr} g'_2 = & 0\ g_{11} + & (\ \ 1)\,g_{21} + & 0\ g_{31} + & 0\ g_{41} \\ + & 0\ g_{12} + & 0\ g_{22} + & 0\ g_{32} + & 0\ g_{42} \\ + & 0\ g_{13} + & 0\ g_{23} + & 0\ g_{33} + & (1)\,g_{43} \\ + & 0\ g_{14} + & 0\ g_{24} + & 0\ g_{34} + & 0\ g_{44} \\ = & g_{21} + & g_{43}, & & \end{array}$$

$$\begin{array}{rrrrr} g'_3 = & 0\ g_{11} + & 0\ g_{21} + & (1)\,g_{31} + & 0\ g_{41} \\ + & 0\ g_{12} + & 0\ g_{22} + & 0\ g_{32} + & (1)\,g_{42} \\ +\,(-1)\,g_{13} + & & 0\ g_{23} + & 0\ g_{33} + & 0\ g_{43} \\ + & 0\ g_{14} + & (-1)\,g_{24} + & 0\ g_{34} + & 0\ g_{44} \\ = & g_{31} + & g_{42} - & g_{13} - & g_{24}, \end{array}$$

$$\begin{array}{rrrrr} g'_4 = & 0\ g_{11} + & 0\ g_{21} + & 0\ g_{31} + & (1)\,g_{41} \\ + & 0\ g_{12} + & 0\ g_{22} + & 0\ g_{32} + & 0\ g_{42} \\ + & 0\ g_{13} + & (-1)\,g_{23} + & 0\ g_{33} + & 0\ g_{43} \\ + & 0\ g_{14} + & 0\ g_{24} + & 0\ g_{34} + & 0\ g_{44} \\ = & g_{41} - & g_{23}. & & \end{array}$$

Substituting the symbolic products from (11) and factoring each expression:

$$\begin{aligned}
g_1' &= e_{11}f_{11} + e_{22}f_{11} + e_{11}f_{22} + e_{22}f_{22} = (e_{11} + e_{22})\,(f_{11} + f_{22}) \\
g_2' &= e_{21}f_{11} + e_{21}f_{22} \qquad\qquad\qquad\quad = \qquad e_{21}\;\,(f_{11} + f_{22}) \\
g_3' &= e_{11}f_{21} + e_{22}f_{21} - e_{11}f_{12} - e_{22}f_{12} = (e_{11} + e_{22})\,(f_{21} - f_{12}) \\
g_4' &= e_{21}f_{21} - e_{21}f_{12} \qquad\qquad\qquad\quad = \qquad e_{21}\;\,(f_{21} - f_{12}).
\end{aligned}$$

The units of one system are represented by $e_{11} + e_{22}$ and e_{21}. The units of the other system are represented by $f_{11} + f_{22}$ and $f_{21} - f_{12}$. The law for the combination of such expressions is $g_{rs}\,g_{qt} = g_{rt}\,\delta_{sq}$. Multiplying out according to this law, we get for the first factor system

$$\begin{array}{c|cc}
 & e_{11} + e_{22} & e_{21} \\
\hline
e_{11} + & e_{11} + 0 & 0 \\
e_{22} & + 0 + e_{22} & + e_{21} \\
e_{21} & e_{21} + 0 & 0.
\end{array}$$

In ordinary notation this system is

$$\begin{array}{c|cc}
 & e_1 & e_2 \\
\hline
e_1 & e_1 & e_2 \\
e_2 & e_2 & 0.
\end{array}$$

For the second factor system, we obtain

$$\begin{array}{c|cc}
 & f_{11} + f_{22} & f_{21} - f_{12} \\
\hline
f_{11} + & f_{11} + 0 & 0 - f_{12} \\
f_{22} & + 0 + f_{22} & + f_{21} - 0 \\
f_{21} - & f_{21} + 0 & 0 - f_{22} \\
f_{12} & - 0 - f_{12} & - f_{11} + 0.
\end{array}$$

In ordinary notation this system is

$$\begin{array}{c|cc}
 & f_1 & f_2 \\
\hline
f_1 & f_1 & f_2 \\
f_2 & f_2 & -f_1.
\end{array}$$

So the given system is the compound of the Cayley two-unit system and the ordinary complex system.

III. Let us take up next the system whose multiplication table is

	h_1	h_2	h_3	h_4	h_5	h_6
h_1	h_1	h_2	h_3	h_4	h_5	h_6
h_2	h_2	0	h_4	0	h_6	0
h_3	h_3	h_4	0	0	0	0
h_4	h_4	0	0	0	0	0
h_5	h_5	h_6	0	0	0	0
h_6	h_6	0	0	0	0	0.

If this is a composite system, it may be either the product of a two-unit system by a three-unit system or the product of a three-unit system by a two-unit system.* Assume for them

	e_1	e_2
e_1	e_{11}	e_{12}
e_2	e_{21}	e_{22}

and

	f_1	f_2	f_3
f_1	f_{11}	f_{12}	f_{13}
f_2	f_{21}	f_{22}	f_{23}
f_3	f_{31}	f_{32}	f_{33}.

Symbolically the compound system has two possible forms

$$\begin{array}{c|cccccc}
 & e_1f_1 & e_1f_2 & e_1f_3 & e_2f_1 & e_2f_2 & e_2f_3 \\
\hline
e_1f_1 & e_{11}f_{11}=g_{11} & e_{11}f_{12}=g_{12} & e_{11}f_{13}=g_{13} & e_{12}f_{11}=g_{14} & e_{12}f_{12}=g_{15} & e_{12}f_{13}=g_{16} \\
e_1f_2 & e_{11}f_{21}=g_{21} & e_{11}f_{22}=g_{22} & e_{11}f_{23}=g_{23} & e_{12}f_{21}=g_{24} & e_{12}f_{22}=g_{25} & e_{12}f_{23}=g_{26} \\
e_1f_3 & e_{11}f_{31}=g_{31} & e_{11}f_{32}=g_{32} & e_{11}f_{33}=g_{33} & e_{12}f_{31}=g_{34} & e_{12}f_{32}=g_{35} & e_{12}f_{33}=g_{36} \\
e_2f_1 & e_{21}f_{11}=g_{41} & e_{21}f_{12}=g_{42} & e_{21}f_{13}=g_{43} & e_{22}f_{11}=g_{44} & e_{22}f_{12}=g_{45} & e_{22}f_{13}=g_{46} \\
e_2f_2 & e_{21}f_{21}=g_{51} & e_{21}f_{22}=g_{52} & e_{21}f_{23}=g_{53} & e_{22}f_{21}=g_{54} & e_{22}f_{22}=g_{55} & e_{22}f_{23}=g_{56} \\
e_2f_3 & e_{21}f_{31}=g_{61} & e_{21}f_{32}=g_{62} & e_{21}f_{33}=g_{63} & e_{22}f_{31}=g_{64} & e_{22}f_{32}=g_{65} & e_{22}f_{33}=g_{66}
\end{array} \tag{12}$$

and

$$\begin{array}{c|cccccc}
 & e_1f_1 & e_2f_1 & e_1f_2 & e_2f_2 & e_1f_3 & e_2f_3 \\
\hline
e_1f_1 & e_{11}f_{11}=g_{11} & e_{12}f_{11}=g_{12} & e_{11}f_{12}=g_{13} & e_{12}f_{12}=g_{14} & e_{11}f_{13}=g_{15} & e_{12}f_{13}=g_{16} \\
e_2f_1 & e_{21}f_{11}=g_{21} & e_{22}f_{11}=g_{22} & e_{21}f_{12}=g_{23} & e_{22}f_{12}=g_{24} & e_{21}f_{13}=g_{25} & e_{22}f_{13}=g_{26} \\
e_1f_2 & e_{11}f_{21}=g_{31} & e_{12}f_{21}=g_{32} & e_{11}f_{22}=g_{33} & e_{12}f_{22}=g_{34} & e_{11}f_{23}=g_{35} & e_{12}f_{23}=g_{36} \\
e_2f_2 & e_{21}f_{21}=g_{41} & e_{22}f_{21}=g_{42} & e_{21}f_{22}=g_{43} & e_{22}f_{22}=g_{44} & e_{21}f_{23}=g_{45} & e_{22}f_{23}=g_{46} \\
e_1f_3 & e_{11}f_{31}=g_{51} & e_{12}f_{31}=g_{52} & e_{11}f_{32}=g_{53} & e_{12}f_{32}=g_{54} & e_{11}f_{33}=g_{55} & e_{12}f_{33}=g_{56} \\
e_2f_3 & e_{21}f_{31}=g_{61} & e_{22}f_{31}=g_{62} & e_{21}f_{32}=g_{63} & e_{22}f_{32}=g_{64} & e_{21}f_{33}=g_{65} & e_{22}f_{33}=g_{66}.
\end{array} \tag{13}$$

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* The systems resulting from the two orders of combination are essentially the same. The uncertainty is one of subscripts in the identification with the symbolic products and may be encountered in factoring any composite system having factors of unequal orders. However the number of trials that may be found necessary is always finite.

The multiplication table of the given system determines the γ's. The substitution of their values in (10) gives

$$\begin{aligned} g_1' &= g_{11} + g_{22} + g_{33} + g_{44} + g_{55} + g_{66} \\ g_2' &= g_{21} + g_{43} + g_{65} \\ g_3' &= g_{31} + g_{42} \\ g_4' &= g_{41} \\ g_5' &= g_{51} + g_{62} \\ g_6' &= g_{61}. \end{aligned}$$

Upon substitution of the symbolic products (12) one obtains

$$\begin{aligned} g_1' &= e_{11}f_{11} + e_{11}f_{22} + e_{11}f_{33} + e_{22}f_{11} + e_{22}f_{22} + e_{22}f_{33} \\ &= (e_{11} + e_{22})(f_{11} + f_{22} + f_{33}) \\ g_2' &= e_{11}f_{21} + e_{21}f_{13} + e_{22}f_{32} \\ g_3' &= e_{11}f_{31} + e_{21}f_{12} \\ g_4' &= e_{21}f_{11} \\ g_5' &= e_{21}f_{21} + e_{21}f_{32} = e_{21}(f_{21} + f_{32}) \\ g_6' &= e_{21}f_{31}. \end{aligned}$$

The units g_2' and g_3' do not factor and therefore the second order of combination (13) must be tried:

$$\begin{aligned} g_1' &= e_{11}f_{11} + e_{22}f_{11} + e_{11}f_{22} + e_{22}f_{22} + e_{11}f_{33} + e_{22}f_{33} \\ &= (e_{11} + e_{22})(f_{11} + f_{22} + f_{33}) \\ g_2' &= e_{21}f_{11} + e_{21}f_{22} + e_{21}f_{33} = e_{21}(f_{11} + f_{22} + f_{33}) \\ g_3' &= e_{11}f_{21} + e_{22}f_{21} = (e_{11} + e_{22})f_{21} \\ g_4' &= e_{21}f_{21} \qquad\qquad\quad = e_{21}(f_{21}) \\ g_5' &= e_{11}f_{31} + e_{22}f_{31} = (e_{11} + e_{22})f_{31} \\ g_6' &= e_{21}f_{31} \qquad\qquad\quad = e_{21}(f_{31}). \end{aligned}$$

This time factors appear and the units of one system are represented by $e_{11} + e_{22}$ and e_{21}. Multiplying out according to the law given above, we obtain for the first factor system

$$\begin{array}{r|cc} & e_{11} + e_{22} & e_{21} \\ \hline e_{11} + & e_{11} + 0 & 0 \\ e_{22} & + 0 + e_{22} & + e_{21} \\ e_{21} & e_{21} + 0 & 0. \end{array}$$

In ordinary notation this system is

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & 0. \end{array}$$

The units of the second factor system are represented by $f_{11}+f_{22}+f_{33}$, f_{21}, and f_{31}. These determine the system

$$\begin{array}{l|ccc}
 & f_{11}+f_{22}+f_{33} & f_{21} & f_{31} \\
\hline
f_{11} & f_{11}+0+0 & 0 & 0 \\
\quad +f_{22} & +0+f_{22}+0 & +f_{21} & +0 \\
\qquad +f_{33} & +0+0+f_{33} & +0 & +f_{31} \\
f_{21} & f_{21}+0+0 & 0 & 0 \\
f_{31} & f_{31}+0+0 & 0 & 0.
\end{array}$$

In ordinary notation this system is

$$\begin{array}{c|ccc}
 & f_1 & f_2 & f_3 \\
\hline
f_1 & f_1 & f_2 & f_3 \\
f_2 & f_2 & 0 & 0 \\
f_3 & f_3 & 0 & 0.
\end{array}$$

IV. The factoring of a composite system of eight units into a two-unit system and a four-unit system* presents no new difficulties and the details of the method may be readily developed from the two preceding examples. By this scheme the octonian system is easily shown to be the compound of the ordinary complex system and the quaternion system.

V. This method at times furnishes curious results. To exhibit this, let us apply the method to the system†

$$\begin{array}{c|cccc}
 & h_1 & h_2 & h_3 & h_4 \\
\hline
h_1 & h_1 & h_2 & h_3 & h_4 \\
h_2 & h_2 & 0 & h_4 & 0 \\
h_3 & h_3 & -h_4 & 0 & 0 \\
h_4 & h_4 & 0 & 0 & 0.
\end{array}$$

Writing out (10) and substituting from (11)

$$\begin{aligned}
g_1' &= g_{11}+g_{22}+g_{33}+g_{44} = e_{11}f_{11}+e_{22}f_{11}+e_{11}f_{22}+e_{22}f_{22} \\
&= (e_{11}+e_{22})(f_{11}+f_{22}) \\
g_2' &= g_{21}+g_{43} = e_{21}f_{11}+e_{21}f_{22} = e_{21}(f_{11}+f_{22}) \\
g_3' &= g_{31}-g_{42} = e_{11}f_{21}-e_{22}f_{21} = (e_{11}-e_{22})f_{21} \\
g_4' &= g_{41} \qquad\;\; = e_{21}f_{21} \qquad\quad = e_{21}(f_{21}).
\end{aligned}$$

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* Of course the four-unit system itself may be factorable.

† Study, Encyklopaedie der Math. Wissen., vol. 1, p. 167 system VIII.

of writing, for inst., S/μ_1 is $= u_2 \cdot \mu_2/\mu_1 + u_3 \cdot \mu_3/\mu_1$. The basis of the pencil consists of $P_1 \ldots . P_7$ and two points on AB. For, indeed, $S/_{AB}$ contains both S_A and S_B.

The relation between S_A and A is evidently this: A line through A intersects S_A in two points completing with $P_1 \ldots . P_7$ the base of a pencil of cubics. In particular, the point that with A and $P_1 \ldots . P_7$ completes that configuration, is the intersection of S_A and the tangent to S_A at A. The other 4 tangents from A to S_A, having their point of contact somewhere else, touch S_A in points whose corresponding point coincides with themselves. Through these points P a cubic is possible having P as doublepoint and containing $P_1 \ldots . P_7$. The curve of the P is of 6th order and contains $P_1 \ldots . P_7$ as doublepoints. It can therefore intersect a cubic through $P_1 \ldots . P_7$ only in 4 points besides $P_1 \ldots . P_7$. Hence the P points on S_A have been completely identified.

S_{P_i} is the curve of S containing P_i as doublepoint. Indeed, it is according to the fundamental relation

$$S_{P_i} \cdot QR = S_Q \cdot P_i R - S_R \cdot P_i Q$$

and both S_Q and S_R contain P_i.

Let any line l be given. Let to one of its points Q the other Q' be joined that combined with it and $P_1 \ldots . P_7$ completes the base of a pencil of cubics. The ∞ straight lines $Q\,Q'$ thus generated will generally belong to a curve of class 3. Indeed if R is an arbitrary point not on l, two points corresponding to each other as Q and Q' will be collinear with R if S_R contains Q and Q'. Hence the points Q on l whose lines $Q\,Q'$ pass through R are the three intersections of l and S_R. If l contains one of the P_i, this reasoning shows that the curve corresponding to l is of class 2. And if l contains two of the P_i, the curve will be a point, namely the point T, belonging to the cubic S_T degenerating into l and the conic through the other 5 points P. This whole reasoning is susceptible of extension to curves of any order, having in the P_i singularities of any kind.

In the geometry of cubics through $P_1 \ldots . P_7$ all relations of ordinary plane geometry have their equivalent. This comes from the fact, that the cubics S_A, S_B, $S_C \ldots .$ are connected by the same equations as the points A, B, $C \ldots .$ themselves. For inst., let A, B, C, D be 4 collinear points such that A and B are harmonically divided by C and D. Then constants α and β will exist, so that

$$C . D = \alpha A^2 + \beta B^2.$$

by S and we shall introduce parameters μ_1, μ_2, μ_3 so that any form of S appears in the shape

$$u_1 \cdot \mu_1 + u_2 \cdot \mu_2 + u_3 \cdot \mu_3$$

where u_1, u_2, u_3 are three forms of S that are linearly independent. Finally, let it be agreed upon that the μ_i shall be treated as variables of a μ plane.

Let then a, b, c be three μ points, μ_1, μ_2, μ_3 as well as S being μ lines. Sa, the composition of S with a, is then a definite cubic form in the plane of the variables x_i, the x plane. For inst., $S/\mu_1/\mu_2$ is u_3. If a, b, c are not on one line, Sa, Sb, Sc will be linearly independent and a relation will exist

$$Sa \cdot g_1 + Sb \cdot g_2 + Sc \cdot g_3 = 0$$

where g_1, g_2, g_3 are lines of the x plane, g_1 the line containing the two residual points of the intersection of Sb and Sc and where g_2, g_3 are similarly determined.

If we transform the Sa, Sb, Sc linearly, then a corresponding identity will exist, where the g_1, g_2, g_3 will experience the corresponding cogredient transformation. Hence the g_1, g_2, g_3 may be interpreted as coordinates of a μ point g.

Let $g = g_1 \cdot a + g_2 \cdot b + g_3 \cdot c$, then $Sa \cdot g\,bc + Sb \cdot g\,ca + Sc \cdot g\,ab = 0$ in virtue of the above identity.

This may also be written $Sg = 0$, i. e. the μ line S composed with the μ point g gives zero as result, no matter what the values of the x_i may be; and thus it is put into evidence, that the relation

$$Sa \cdot g\,bc + Sb \cdot g\,ca + Sc \cdot g\,ab = 0$$

will hold good no matter what μ points a, b, c may be chosen.

$g\,ab$ evidently intersects Sa, in virtue of the fundamental relation, besides in the two residual points of its intersection with Sb, also in a point on $g\,ca$. Thus there is a certain x point A on Sa, in which $g\,ab$ intersects that curve no matter how b may be chosen. A is coresidual to the two other points of intersection of $g\,ab$ with Sa, and, these lying on Sb, residual to $P_1 \ldots P_7$.

a being given, A is determined, because Sa is uniquely determined: $g\,ab$ is the line joining the two points A, B corresponding to a and b. In the determination of the μ space there is so much freedom, that we might identify simply a with A, $g\,ab$ with AB. Thus the original identity reads

$$S_A \cdot BC + S_B \cdot CA + S_C \cdot AB = 0$$

$S/_{AB}$ is a μ point, and a pencil of cubics in the x plane. In the above manner

Multiplication is indicated by a dot. $A . B$ denotes, for inst., the product of the points, or linear forms, A and B.

The notation just described is perhaps a trifle simpler and more expressive than the ordinary notation of the invariant calculus, but is, on the whole, very little different from it.

The author has in the examples chosen made use of an auxiliary space, which he has called the λ space. Inasmuch as frequently sets of equations of the type

$$\begin{aligned} a_1 . u_1 + a_2 . u_2 + \dots &= 0 \\ b_1 . u_1 + b_2 . u_2 + \dots &= 0 \end{aligned}$$

are discussed, he has written them in one line

$$(\lambda_1 . a_1 + \lambda_2 . b_1 + \dots) . u_1 + (\lambda_1 . a_2 + \lambda_2 . b_2 + \dots) . u_2 + \dots = 0$$

and afterwards treated the expressions $\lambda_1 a_1 + \dots$ as linear forms, i. e. points, of the above mentioned λ space. This way of proceeding, though not necessary, seems useful for the purpose of simplifying the calculations that would otherwise be beyond control. It allows, for inst., the advantage of the use of well-known identical relations of the invariant calculus, and it is the distinctive character of the method studied in the paper that the coefficients of these identities are not mere numbers, but forms in the original space, called the x space.

The curve whose equation is $f = 0$ is, in what follows, frequently without further comment denoted with f. This notation seemed almost necessary in a paper where identical relations, such as

$$u_1 . v_1 + u_2 . v_2 + \dots = 0$$

had to be discussed. In any case this way of denoting curves and geometrical formations of any kind has its advantages. It permits to identify an irreducible formation directly with the corresponding prime modulus.

The author's paper twice referred to in what follows, "Zur Theorie der Moduln und Ideale," appeared in the Mathematische Annalen, 1904.

As a first example let a case be considered where one definite syzygetic relation exists between three plane forms. The three forms may be three cubics that have 7 points $P_1, P_2 \dots, P_7$ in common, of which we suppose that no 6 of them are on one conic. The system of cubics through $P_1 \dots P_7$ will be denoted

A New Method in Geometry.

By E. Lasker.

INTRODUCTION.

The following lines are based on the theory of moduli, the youngest branch of mathematical science. A method of research in intimate connection with that theory and applicable to algebraical, nay, even analytical formations of any kind, will here be discussed and illustrated. The examples chosen to explain the method, and to show its usefulness, are of a simple nature, and do not require the reader to be acquainted with more of the theory of moduli than is contained in the "Fundamentaltheorem" of Noether.

The method which is the subject of this paper consists in the treatment of formations or configurations of such by means of the syzygetic relations that connect the basic forms of the modulus or moduli corresponding to the configurations. The linear system of such relations of a given order is studied by treating this linear system as an auxiliary space.

The author has used the notation of his "Essay on the geometrical calculus," published in the Proceedings of the London Mathematical Society, 1895 and 1896. This notation may be briefly explained for the plane. If A, B, C are linear forms or points, then ABC denotes their determinant, AB the line joining A and B, and $AB \doteq -BA$. If a, b, c are linear forms subject to contragredient transformations or lines, $a/b/c$ denotes their determinant and a/b the point of intersection of a and b. Any relation between points, such as, say

$$A^2 + 2B^2 = 3C^2 + 9D^2$$

if true, expresses no more nor less than that, if the points-symbols used are simultaneously composed with an arbitrary line ($l = EF$; $Al = AEF$) the relation is true (so that in the above instance

$$(Al)^2 + 2(Bl)^2 = 3(Cl)^2 + 9(Dl)^2).$$

then the number x is called a left-hand divisor of zero. Similarly if

$$\Delta'_y \equiv \left| \begin{matrix} \sum_{i_2} y_{i_2} \, \gamma_{i_1 i_2 i_3} \\ i_1, i_3 = 1, \ldots, n \end{matrix} \right| \equiv 0, \tag{16}$$

then the number y is called a right-hand divisor of zero. The substitution of $\mu = 0$ in (4) gives a form which is evidently (15) and consequently Δ_x is the absolute term in that type of characteristic equation. Similarly Δ'_y is the absolute term in the other type of characteristic equation suggested in a previous foot note (§2).

If the absolute term of the characteristic equation of the general number of a system vanishes, then every number of that system is a left-hand divisor of zero. It is known that in every system except the real, the ordinary complex and the quaternion, special numbers can be found for which the characteristic equation has no absolute term and such numbers are divisors of zero.

From the theory of equations it is plain that* ${}_E\Delta_x$ is the product of the n roots, μ_i, of the characteristic equation of a number of the system E and that ${}_F\Delta_x$ is the product of the r roots, ν_j, of the characteristic equation of a number of the system F. Then the absolute term of the characteristic equation of their compound number is the product of the nr roots, $\mu_i \nu_j$, and in this product each root μ_i occurs r times and each root ν_j occurs n times. Therefore

$$ {}_{EF}\Delta_x = ({}_E\Delta_x)^r \cdot ({}_F\Delta_x)^n. \tag{17}$$

From (17) it is evident that if either of the factor numbers is a divisor of zero, then the compound number must be a divisor of zero.

If the absolute term of the characteristic equation of a general composite number of a composite system vanishes, then every composite number of this system is a left-hand divisor of zero and in the factor systems every number of one (or perhaps of both) is a left-hand divisor of zero.

If the general composite number is not a divisor of zero, it may still be that there are special composite numbers which are divisors of zero (that is, while ${}_{EF}\Delta_x$ does not vanish identically, it may vanish for special values of the x's). In this case, it follows as above that at least one of the factors of the composite number is a divisor of zero.

THE UNIVERSITY OF COLORADO, *June*, 1906.

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* The first subscript indicates the system under consideration.

The compound system is

	$e_1 f_1$	$e_2 f_1$	$e_3\ f_2$	$e_4\ f_2$	$e_4 f_1$	$e_3 f_1$	$e_2\ f_2$	$e_1\ f_2$
$e_1 f_1$	$e_1 f_1$	$e_2 f_1$	$e_3\ f_2$	$e_4\ f_2$	$e_4 f_1$	$e_3 f_1$	$e_2\ f_2$	$e_1\ f_2$
$e_2 f_1$	$e_2 f_1$	$-e_1 f_1$	$e_4\ f_2$	$-e_3\ f_2$	$-e_3 f_1$	$e_4 f_1$	$-e_1\ f_2$	$e_2\ f_2$
$e_3 f_2$	$e_3 f_2$	$-e_4 f_2$	$e_1 . 0$	$-e_2 . 0$	$-e_2 f_2$	$e_1 f_2$	$-e_4 . 0$	$e_3 . 0$
$e_4 f_2$	$e_4 f_2$	$e_3 f_2$	$e_2 . 0$	$e_1 . 0$	$e_1 f_2$	$e_2 f_2$	$e_3 . 0$	$e_4 . 0$
$e_4 f_1$	$e_4 f_1$	$e_3 f_1$	$e_2\ f_2$	$e_1\ f_2$	$e_1 f_1$	$e_2 f_1$	$e_3\ f_2$	$e_4\ f_2$
$e_3 f_1$	$e_3 f_1$	$-e_4 f_1$	$e_1\ f_2$	$-e_2\ f_2$	$-e_2 f_1$	$e_1 f_1$	$-e_4\ f_2$	$e_3\ f_2$
$e_2 f_2$	$e_2 f_2$	$-e_1 f_2$	$e_4 . 0$	$-e_3 . 0$	$-e_3 f_2$	$e_4 f_2$	$-e_1 . 0$	$e_2 . 0$
$e_1 f_2$	$e_1 f_2$	$e_2 f_2$	$e_3 . 0$	$e_4 . 0$	$e_4 f_2$	$e_3 f_2$	$e_2 . 0$	$e_1 . 0$

or

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_3
e_2	e_2	$-e_1$	e_4	$-e_3$	$-e_6$	e_5	$-e_8$	e_7
e_3	e_3	$-e_4$	0	0	$-e_7$	e_8	0	0
e_4	e_4	e_3	0	0	e_8	e_7	0	0
e_5	e_5	e_6	e_7	e_8	e_1	e_2	e_3	e_4
e_6	e_6	$-e_5$	e_8	$-e_7$	$-e_2$	e_1	$-e_4$	e_3
e_7	e_7	$-e_8$	0	0	$-e_3$	e_4	0	0
e_8	e_8	e_7	0	0	e_4	e_3	0	0.

Our given system is a sub-system of the eight-unit system. This system is peculiar. Let $j = 1, 2, 3, 4$ and $k = 5, 6, 7, 8$, then

$$e_{j_1} e_{j_2} = e_{j_3}, \quad e_{j_1} e_{k_2} = e_{k_3}, \quad e_{k_1} e_{j_2} = e_{k_3}, \text{ and } e_{k_1} e_{k_2} = e_{j_3}.$$

§5.—DIVISORS OF ZERO.

The product of $x = \sum_{i_1} x_{i_1} e_{i_1}$ and $y = \sum_{i_2} y_{i_2} e_{i_2}$ in the system E is

$$xy = \sum_{i_1 i_2 i_3} x_{i_1} y_{i_2} \gamma_{i_1 i_2 i_3} e_{i_3} = \sum_{i_3} z_{i_3} e_{i_3}. \tag{14}$$

If

$$\Delta_x \equiv \left| \begin{matrix} \sum_{i_1} x_{i_1} \cdot \gamma_{i_1 i_2 i_3} \\ i_2, i_3 = 1, \ldots, n \end{matrix} \right| \equiv 0, \tag{15}$$

Hence, since all relations between the points are conserved in the corresponding cubics, we have

$$S_C \cdot S_D = \alpha S_A^2 + \beta S_B^2.$$

Interpreting this result for the points $P_1 \ldots P_7$, Q, R common to S_A, S_B, S_C, S_D, we have the proposition: If A, B, C, D are in harmonic situation, the tangents of S_C and S_D at any one of the P_i, Q, R divide the tangents of S_A and S_B at these points harmonically.

By the same reasoning the connection between the six points that are the intersections of 4 straight lines gives a similar relation between the tangent lines at P_i of the cubics S belonging to the six points. And this principle may be used with every identical relation between points or lines of a plane. The rule is in fact susceptible of yet wider extension as its demonstration makes plain without difficulty.

Wherever between a set of forms a single syzygetic relation exists, as above, the introduction of the μ space is advisable. In what follows we shall however do away with this expedient, in order not to confuse by the introduction of two or more auxiliary spaces. He who can handle operations in various sets of variables with ease will probably be able to shorten much of the work done in what follows. But this capacity is a rare accomplishment. Let u, v, w be forms of the 4th order which have eight points in common and such that no two of them have an infinity of points in common.

The eight points common to $u = 0$, $v = 0$, $w = 0$ may be denoted by P_1, $P_2 \ldots P_8$ and it is supposed that they do not lie on a conic. The two curves $u = 0$, $v = 0$ will, generally speaking, have 8 more points in common; or, to be accurate, the modulus (u, v) will comprise in all 16 Noetherian conditions. If $f = 0$ is any curve containing P_1, $P_2 \ldots P_8$, then any form F, such that $f \cdot F$ belongs to the modulus (u, v), must satisfy 8 conditions, to which we briefly refer as the residual conditions of (u, v). It is not accurate to say that F must contain 8 determinate points in order to satisfy the above relation, because this expresses the truth only when u and v have 16 distinct points of intersection and there may be coincidences. But to simplify the manner of expression we shall assume that u and v are not in contact, and, should in a given case this not be so, we shall understand that the Noetherian conditions of the F-modulus will then take the place of the coinciding points. Nor shall this remark be restricted to the case under discussion. In all that follows we shall disregard coincidences of points unless otherwise stated, because the complication thus arising is without influence on the line of reasoning and easy to dissolve by the method of limits, as demonstrated in the paper "Zur Theorie der Moduln und Ideale."

$u = 0$, $v = 0$ will then have 8 residual points in common. These will not lie on a conic, for, from the proposition of Cayley in respect to the intersection of two plane curves it easily follows that, if 8 points of the intersection of two quartics are on a conic then the residual 8 points must likewise be on a conic. And we know that $P_1 \ldots P_8$ are not on a conic.

Let now c_1, c_2 be two forms of third order, such that $c_1 = 0$ and $c_2 = 0$ contain the 8 residual points of $u = 0$, $v = 0$. $w \cdot c_1 = 0$ and $w \cdot c_2 = 0$ will then contain all the 16 points of intersection of $u = 0$ and $v = 0$, and therefore two relations will exist

$$u a_1 + v b_1 + w c_1 = 0$$
$$u a_2 + v b_2 + w c_2 = 0$$

where a_1, a_2, b_1, b_2, c_1, c_2 are cubic forms.

Multiplying the first of these equations by c_1, the second by c_2, we obtain by subtraction

$$u(a_1 c_2 - a_2 c_1) + v(b_1 c_2 - b_2 c_1) = 0,$$

hence

$$a_1 c_2 - a_2 c_1 = - v \cdot t$$
$$b_1 c_2 - b_2 c_1 = u t$$

where t is a form of 2nd order. By eliminating from the two equations u we obtain also

$$b_1 a_2 - a_1 b_2 = w t.$$

The interpretation of these equations gives the following results:

From $u a_1 + v b_1 + w c_1 = 0$, as u, v, w have $P_1 \ldots P_8$ in common, it follows that c_1 contains the 8 residual points of (u, v), b_1 those of (u, w), a_1 those of (v, w). c_1 and v have 12 points in common, hence 4 of these points lie on a_1. But c_1 and a_1 have 9 points in common. Consequently a_1, b_1 and c_1 have 5 common points. These, as a look on the three last equations shows, evidently lie on t.

It may equally be inferred that all points common to (a_1, a_2) not on v and w must be on t. Therefore the 9th point of intersection of the two cubics a_1 and a_2 is on t. The same applies to the 9th point of intersection of (b_1, b_2) and (c_1, c_2).

Let us write the two above equations

$$(\lambda_1 a_1 + \lambda_2 a_2) u + (\lambda_1 b_1 + \lambda_2 b_2) v + (\lambda_1 c_1 + \lambda_2 c_2) w = 0$$

where λ_1, λ_2 are indeterminatae. If λ_1, λ_2 are constants, so determined, that $\lambda_1 a_1 + \lambda_2 a_2$ contains point P_1, then $(\lambda_1 a_1 + \lambda_2 a_2) u$ will have P_1 as doublepoint. v and w not having contact at P_1, it follows that also $\lambda_1 b_1 + \lambda_2 b_2$ and $\lambda_1 c_1 + \lambda_2 c_2$ must contain P_1. Hence the equation $a u + b v + c w = 0$ is satisfied, if a is the cubic containing the 8 residual points of (v, w) and P_1, b the cubic containing the 8 residual points of (w, u) and P_1, and c is correspondingly determined. a, b, c will then have 5 points in common not on u, v, w, and the conic through them is t. For this construction of t any one of the 8 points $P_1 \ldots P_8$ may be utilized. The most general solution of

$$a u + b v + c w = 0$$

where a, b, c are cubics, is then attained by taking an arbitrary point P on t, and constructing a, b, c through their residual 8 points and P.

t is a concomitant of u, v, w which is multiplied by a factor only when u, v, w are subject to a linear transformation, when, for inst., u is replaced by $\alpha u + \beta v + \gamma w$. u, v, w define a linear system S of quartics through $P_1 \ldots P_8$ and $t = 0$ is the locus of the point that, with 8 points forming the residual intersection of any two curves of S, completes the configuration of 9 points common to two cubics.

All this may easily be extended to three curves of order n. Thus we may announce: If u, v, w are three curves of nth order having $\frac{1}{2} n (n - 1) + 2$ points P_i in common not situated on a curve of order $n - 2$, then two equations exist

$$u a_1 + v b_1 + u c_1 = 0$$
$$u a_2 + v b_2 + w c_2 = 0$$

where a_1, b_1, c_1, a_2, b_2, c_2 are forms of order $n - 1$. a, b, c have $\frac{1}{2} (n - 1 . n) - 1$ points in common that determine a curve $t = 0$ of order $n - 2$. u, v, w determine a linear system S and any two curves of S intersect in the P_i and in a residual group of $\frac{1}{2} n (n + 1) - 2$ points, which forms part of the base of a pencil of curves of order $n - 1$. The remaining $\frac{(n - 2)(n - 3)}{2}$ basic points of this pencil are always situate on t.

Further, if u', v', w' are any three linearly independent members of S, the residual pointgroups corresponding to (u', v'), (v', w'), (w', u') are such that triplets of curves of order $(n-1)$ through them and any one of the P_i intersect on $t = 0$.

Let us return now to the original case of $n = 4$. a_1, a_2 were curves of 3d order through the 8 residual points of (v, w). Let A be the 9th point of intersection of a_1, a_2, and let β and γ be such constants that

$$\gamma v - \beta w$$

contains A. Then this curve will contain all points of intersection of a_1, a_2, hence linear forms p and q will exist such that

$$\gamma v - \beta w = p a_1 + q a_2 .$$

$p a_1 + q a_2$ will therefore contain $P_1 \ldots . P_8$. Reverting to the argument above, referring to $\lambda_1 a_1 + \lambda_2 a_2$ containing one of the points $P_1 \ldots . P_8$, it is clear that the same line of reasoning shows $p b_1 + q b_2$ as well as $p c_1 + q c_2$ to contain the points $P_1 \ldots . P_8$. Hence constants α, α', β', γ' will exist such that

$$p b_1 + q b_2 = \alpha w - \gamma' u$$

and

$$p c_1 + q c_2 = \beta' u - \alpha' v.$$

But identically

$$u (p a_1 + q a_2) + v (p b_1 + q b_2) + w (p c_1 + q c_2) = 0$$

therefore

$$u (\gamma v - \beta w) + v (\alpha w - \gamma' u) + w (\beta' u - \alpha' v) = 0$$

and it follows $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$.

Moreover, since identically

$$\alpha (\gamma v - \beta w) + \beta (\alpha w - \gamma u) + \gamma (\beta u - \alpha v) = 0$$

we have

$$\alpha (p a_1 + q a_2) + \beta (p b_1 + q b_2) + \gamma (p c_1 + q c_2) = 0$$

and

$$\alpha a_1 + \beta b_1 + \gamma c_1 = p \,.\, T$$
$$\alpha a_2 + \beta b_2 + \gamma c_2 = - q \,.\, T$$

where T is a form of 2nd order, which, from the fact that a_1, b_1, c_1 have 5 points in common with $t = 0$, can be easily shown to be identical with t.

p and q intersect in the point that together with $P_1 \ldots . P_8$ is the base of a pencil of cubics. Generally the proposition holds that the quartic which contains the 16 points of the base of a quartic pencil and the 9th point completing with

8 of them the base of a cubic pencil also contains the 9th point completing with the 8 others the base of a cubic pencil.

To show this, let 1....8 and 9....16 be the 16 points common to two quartics; let further I be the point that jointly with 1....8 makes the base of a cubic pencil, and let II be the corresponding point for the set 9....16. The quartic through 1....16 and I may be u, another through 1....16, but neither through I nor II, may be v. Also let A, B be cubics through 1....8 I. Then

$$u = A\alpha - B\beta$$

where α, β are suitably determined linear forms. Also, if p and p' are lines through I

$$vp = A\gamma - B\delta$$
$$vp' = A\gamma' - B\delta'$$

where γ, δ, γ', δ' are forms of 2nd order.

From the two last equations

$$v(p\delta' - p'\delta) = A(\gamma\delta' - \gamma'\delta)$$

hence $$p\delta' - p'\delta = A\varepsilon$$

and $$\gamma\delta' - \gamma'\delta = v\varepsilon$$

also $$p\gamma' - p'\gamma = B\varepsilon$$

where ε is a constant.

We have $u\delta - vp\beta = (\alpha\delta - \beta\gamma)A$.

The points 9....16 lie therefore on $\alpha\delta - \beta\gamma$ as well as on $\alpha\delta' - \beta\gamma'$. Being cubics these forms also contain the point II. But we have identically

$$u(\gamma\delta' - \gamma'\delta) + vp(\gamma'\beta - \delta'\alpha) + vp'(\alpha\delta - \beta\gamma) = 0$$

hence $$u\varepsilon + p(\gamma'\beta - \delta'\alpha) + p'(\alpha\delta - \beta\gamma) = 0$$

u contains therefore II.

Applying this proposition to the equations evolved previously, we conclude that $\alpha w - \gamma u$, $\beta u - \alpha v$, $\gamma v - \beta w$ contain the point P that with $P_1 \ldots P_8$ completes the base of a cubic pencil. These three quartics evidently belong to one pencil, and the 7 points they have in common besides $P_1 \ldots P_8$ P lie on $t = 0$. To construct t it is therefore only necessary to find the base of the pencil of curves of the system S that contain point P.

The previously deduced laws may be easily extended to forms of higher orders. It is only necessary to interpret the set of identities already made use of for forms of higher orders in order to obtain the corresponding laws concerning them. To give an instance, let in the set of equations used for the demonstration of the proposition concerning a pencil of quartics u be a form of nth order ($n > 4$). Then we obtain immediately the new proposition: Any form of nth order intersecting a quartic v in $4n$ points and containing one, P, that with 8 of these points completes the base of a cubic pencil, also contains a set of $(n-3)^2$ points that in conjunction with the other $4(n-2)$ points of intersection makes the base of a pencil of $(n-1)^{\text{th}}$ order. In addition it may be shown that this set of $(n-3)^2$ points is the base of a pencil of order $(n-3)$ (namely of the pencil containing α and β).

Let us now attack a case where more than two syzygetic relations obtain, for instance that of three cubics having six points in common. If u, v, w are these cubics, and if the six points $P_1 \ldots P_6$ common to them are not on a conic, three relations will exist

$$\begin{aligned} a_1 u + b_1 v + c_1 w &= 0 \\ a_2 u + b_2 v + c_2 w &= 0 \\ a_3 u + b_3 v + c_3 w &= 0 \end{aligned}$$

where the a_i, b_i and c_i are conics. For the existence of a relation

$$a u + b v + c w = 0$$

it is only required that $a = 0$ should contain the three residual points of the intersection of (v, w), and, a being a conic chosen in conformity with this condition, b and c are uniquely determined. Here the number of forms is considerable and, to deal with them efficiently, it is advisable to write the three relations in one line

$$0 = (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) u + (\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) v + (\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3) w$$

and to interpret $\lambda_1, \lambda_2, \lambda_3$ as indeterminate coordinates of a point in a λ plane. Finally we write

$$\begin{aligned} \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 &= a \\ \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 &= b \\ \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 &= c \end{aligned}$$

so that a, b, c are points in the λ plane and of the 2nd order in the original plane which, for brevity, we designate the x plane.

The equation $a u + b v + c w = 0$ shows that the three λ points a, b, c are always collinear. Composing the equation with a, we obtain

$$a b . v + a c . w = 0$$

where $a b$ denotes the matrix composed of a and b and where multiplication is indicated by the point. From this relation it follows that $a b$ is divisible by w and ac by v. A form g will therefore exist such that

$$a b = g . w$$
$$c a = g . v$$

and similarly

$$b c = g . u$$

where g is a line in the λ plane and in the x plane is of the 1st order.

Let l be any λ line. Then al, bl, cl, the compositions of the λ points a, b, c with l, are magnitudes in the λ space, and conics in the x plane.

From $a l . u + b l . v + c l . w = 0$ it is clear by reasoning analogous to that previously used, that $a l$, $b l$ and w have three, and therefore $a l$, $b l$ and $c l$ have one point in common. This point lies on the line $g/l/m$, m denoting any other λ line. $g/l/m$ contains also the 4th point of intersection of $(a l, a m)$, $(b l, b m)$, $(c l, c m)$. This fact is put in evidence by the identity

$$a l . b m - a m . b l = a b / l / m = w . g / l / m$$

$a l$ and $a m$ have 3 points in common with w, hence their 4th point of intersection lies on $g/l/m$.

Let L be any λ point. $g L$ denotes then a definite x line, and $a L$, $b L$, $c L$ pencils of conics. $a L$ always contains the three residual points of (v, w); therefore only the 4th point of the base of the pencil $a L$ is variable. It lies on $g L$, and the same is true of the 4th point of the pencils $b L$, $c L$. Indeed, it is identically

$$a L N . b M N - a M N . b L N = a b N . L M N = w . g N . L M N$$

where N is any λ point. Hence $a L N$ and $a M N$ intersect apart from the three points of intersection on w, on $g N$. Similarly the point common to $a L N$, $b L N$, $c L N$ is the point of intersection of the lines $g L$ and $g N$.

To any point A of the x plane correspond two others B and C in this fashion: Through A a pencil of conics al and am is determined. To it correspond pencils $(b l, b m)$ and $(c l, c m)$, whose 4th point of intersection is B and C respectively. A, B, C are in a straight line, namely $g/l/m$. al, bl, cl and am, bm, cm intersect on the same straight line.

The correspondence between A, B, C is therefore such that if A moves on any conic through the 3 basic points of the system a, B and C will move on related conics, while the line A, B, C will revolve round the point common to the three conics in question.

If A coincides with one of the points P_i, then B and C will also coincide with it. This follows at once from the fundamental identity when it is assumed that a contains P_i.

To two conics of the set a through P_1 correspond two conics of the sets b and c through P_1 and each triplet of them determines a point of intersection. The two points of intersection thus derived and P_1 are collinear.

Hence a certain line p_1 passes through P_1, and similarly 5 other lines p_2; $p_3 \ldots p_6$ pass through P_2, P_3, P_4, P_5, P_6 such that to each conic of the set a through P_i correspond conics of the set b and c through P_i intersecting on p_i. Consequently the conics of the sets a, b, c through P_i and P_j intersect in the point of intersection of p_i and p_j. A very curious net of intersections is thus generated.

The fact that for each position of the λ line l al, bl, cl contain, each, three fixed points, leads to this proposition: Besides g three λ lines α, β, γ exist, which are in the x plane of first order and which composed with the λ points a, b, c give zero. With other words, if a_1, a_2, a_3 are the components of the λ point a two identities exist

$$a_1 g_1 + a_2 g_2 + a_3 g_3 = 0, \qquad a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 = 0,$$

where g_1, g_2, g_3 are the coefficients of the λ line g, α_1, α_2, α_3 those of another λ line α. a is then a λ point common to g and α, hence

$$a = g/\alpha, \qquad b = g/\beta, \qquad c = g/\gamma.$$

And from $au + bv + cw = 0$ it follows

$$(u\alpha + v\beta + w\gamma)/g = 0.$$

$u \,.\, \alpha + v \,.\, \beta + w \,.\, \gamma$ is therefore congruent with g, i. e. a multiple of g. We may put it $= - t \,.\, g$, where t is a form of 3d order in the x plane, a number in the λ plane. Thus we have

$$u \,.\, \alpha + v \,.\, \beta + w \,.\, \gamma + t \,.\, g = 0,$$

and $$\alpha\beta\gamma = t, \qquad \beta\gamma g = u, \text{ etc.}$$

If L is any λ point, then αL, βL, γL, $g L$ correspond to each other according to the rules of linear transformations, i. e. to any given g line corresponds one α, β, γ line, and vice versa; and to a g line through a given point correspond α, β, γ lines through dependent (correspondent) points.

Now from $g/\alpha = a$ and the identity

$$g L \,.\, \alpha M - g M \,.\, \alpha L = g/a/L M = a L M$$

it is evident that $g L$ and $g M$ as well as $g L$ and αL intersect on $a L M$. Consequently the intersection of $g L$ and $g M$ is again found to be the point common to $a L M$, $b L M$, $c L M$; and the 4th point of intersection of $a L M$ and $a L N$ (formerly called A) is on $g L$ and αL. The triangle of self-corresponding lines of $g L$, αL is evidently that formed by the three points residual to the intersection of (v, w). The linear correspondences of the x plane, characterized by $g L$, αL, βL, γL, may therefore be constructed as follows: Let the three residual triangles of (u, v), (v, w), (w, u) be the self-corresponding ones of three linear transformations; let further to the line p_1 correspond three arbitrarily selected distinct lines α', β', γ' through P_1. Then three correspondences α, β, γ of the plane are thereby determined, such that any line of the plane is cut by its corresponding α line in the point previously called A, etc., and that α, β, γ lines corresponding to the same (g) line intersect in the same point P only, when P is on a cubic $t = 0$ that contains the six points P_1, P_2, P_3, P_4, P_5, P_6.

All this may again be immediately extended to suitably restricted forms of higher orders, for inst. to three quartics having 9 points in common. The matter of generalization becomes simply a question of counting the number of points of intersection, the order numbers of the various curves introduced and the number of constants of these curves. It is not difficult to extent this method to three plane curves of any orders having any number of common points and therefore any number of syzygetic relations. We shall now enlarge the scope of the work by considering the relations of 4 given plane forms.

As a first instance take the case of 4 conics u_1, u_2, u_3, u_4 having no common point and which are linearly independent. Their linear system may be S. Let g_1, g_2, g_3, g_4 be forms of 1st order, then two independent relations of the type

$$g_1 u_1 + g_2 u_2 + g_3 u_3 + g_4 u_4 = 0$$

will exist. That there will be at least 2 follows from the consideration of the cubics through the 8 points (u_1, u_2), (u_3, u_4), which evidently must be expressible in the way $g_1 u_1 + g_2 u_2$ as well as $g_3 u_3 + g_4 u_4$. That there will be no more than two is clear from the fact that these 8 points cannot lie on one conic (or u_1, u_2, u_3, u_4 would be linearly dependent). Hence g_1, g_2, g_3, g_4 will be points on a λ line, or homographics, whose vertices are somewhere in the x plane.

Composing the identity $g_1 \cdot u_1 + g_2 \cdot u_2 + g_3 \cdot u_3 + g_4 \cdot u_4 = 0$ with g_4, we have

$$g_1 g_4 \cdot u_1 + g_2 g_4 \cdot u_2 + g_3 g_4 \cdot u_3 = 0.$$

Hence constants $\alpha_1, \alpha_2, \alpha_3$ will exist such that

$$\begin{aligned} g_1 g_4 &= \alpha_2 \cdot u_3 - \alpha_3 \cdot u_2 \\ g_2 g_4 &= \alpha_3 \cdot u_1 - \alpha_3 \cdot u_3 \\ g_3 g_4 &= \alpha_1 \cdot u_2 - \alpha_2 \cdot u_1 . \end{aligned}$$

It also follows that

$$\alpha_1 \cdot g_1 g_4 + \alpha_2 \cdot g_2 g_4 + \alpha_3 \cdot g_3 g_4 = 0.$$

$\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2 + \alpha_3 \cdot g_3$ is therefore a numerical multiple of g_4. Consequently a number α_4 exists such that

$$\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2 + \alpha_3 \cdot g_3 + \alpha_4 \cdot g_4 = 0.$$

We shall now interpret these equations. g_1, g_2, g_3, g_4 are homographic pencils of lines through points that will be called A_1, A_2, A_3, A_4. $g_1 g_4, g_2 g_4, g_3 g_4$ are conics having 4 points in common, namely A_4 and say E, F, G. $g_1 g_4$ contains A_1, A_4 and the 4 points (u_2, u_3). From the identity

$$g_1 g_2 \cdot g_4 l + g_2 g_4 \cdot g_1 l + g_4 g_1 \cdot g_2 l = 0$$

where l is any λ point, it follows that also $g_1 g_2$ contains E, F, G. These three points are therefore common to the six conics $g_1 g_2 \cdots g_3 g_4$.

The pencil of cubics $u_1 \cdot g_1 + u_2 \cdot g_2$ contains the 4 points (u_1, u_2) and as the original identity shows, also the four (u_3, u_4). The pencil passes therefore also through another point $B_{1,2} = B_{3,4}$. Let $B_{1,3} = B_{2,4}$ and $B_{2,3} = B_{1,4}$ be similarly determined. Since $u_1 \cdot g_1 + u_2 \cdot g_2$ contains $B_{1,2}$, also $(u_1 \cdot g_1 + u_2 \cdot g_2)\, g_2 = u_1 \cdot g_1 g_2$ will. Hence $B_{1,2}$ is situate on both $g_1 g_2$ and $g_3 g_4$. The points of intersection of the six conics $g_1 g_2 \cdots g_3 g_4$ are herewith completely laid down.

$\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2$ is a pencil whose vertex lies on $(\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2)\, g_2 = \alpha_1 \cdot g_1 g_2$. It also lies on $g_3 g_4$. But its vertex is neither E nor F or G, because it generally does not lie on $g_1 g_3$, which may be most easily shown by the analysis of a particular example. Such an example is most readily obtained by starting inversely from 4 linearly dependent pencils $g_1 \dots g_4$ and constructing 4 forms u_1, u_2, u_3, u_4 according to the above identities. It then follows that the vertex of $\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2$ must be $B_{1,2}$. If l is any λ point $\alpha_1 \cdot g_1 l + \alpha_2 \cdot g_2 l$ in virtue of the relation between g_1, g_2, g_3, g_4 is the line joining the intersection of $g_1 l$ and $g_2 l$ with that of $g_3 l$ and $g_4 l$. This line, with varying l, revolves round $B_{1,2}$.

E, F, G remain invariant when u_1, u_2, u_3, u_4 are subject to linear transformations. Starting with *any* 4 forms of S, E, F, G will therefore remain the same. But S contains 4 squares of linear forms and it can be shown without difficulty that E, F, G are the corners of the diagonal triangle of the complete quadrangle of lines whose squares belong to S.

Summarizing, we obtain a proposition as follows: Let 4 conics u_1, u_2, u_3, u_4 not containing a common point nor linearly dependent be arbitrarily given. The 8 points (u_1, u_2) and (u_3, u_4) determine the point $B_{1,2}$ completing the base of a pencil of cubics through them. $B_{1,3}$ and $B_{2,3}$ are similarly constructed. The conic through $B_{1,2}$ and (u_3, u_4) has with that through $B_{1,2}$ and (u_1, u_2) three points E, F, G in common which, with 4 points common to any two conics of the system S, always lie on one conic. The conics through E, F, G and (u_i, u_j) may be denoted $g_i\, g_j$. Then $g_1\, g_4$, $g_2\, g_4$, $g_3\, g_4$ have besides E, F, G another point A_4 in common, etc. Through A_1, A_2, A_3, A_4 a single infinity of lines g_1, g_2, g_3, g_4 will pass whose 6 points of intersection will lie on the 6 above conics $g_1 g_2 \dots g_3 g_4$ and the sides of whose diagonal triangles revolve round $B_{1,2}$, $B_{1,3}$, $B_{2,3}$. It also easily follows that each corner of these diagonal triangles moves upon a conic through E, F, G (the point of intersection of the lines (g_1/g_2), $(g_3/g_4) \,|\, (g_1/g_3)$, (g_2/g_4) for inst. moves upon the conics $(\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2)$, $(\alpha_1 \cdot g_1 + \alpha_3 \cdot g_3)$).

The process just made use of, if applied to a set of forms of nth order, will lead to very remarkable results. But the calculation becomes somewhat complex when n is large. The principles involved may however be well explained in the case $n = 3$. Let then u_1, u_2, u_3, u_4 be 4 cubics that are linearly independent, have no point in common and such that through the 18 points (u_1, u_2) and (u_3, u_4) only the minimum number, namely three linearly independent quintics can pass.

Let v_1, v_2, v_3, v_4 be conics, suitably determined, then three independent relations will exist

$$(1) \qquad u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 + u_4 \cdot v_4 = 0$$

and the λ space will therefore be a plane. It follows $v_1 v_2 v_3$ is divisible by u_4,

(2)
$$v_1 v_2 v_3 = u_4 . w$$

where w is a x form of 3d order, a number in the λ plane. Similarly

(2)
$$v_4 v_2 v_3 = - u_1 . w, \text{ etc.}$$

Composing the fundamental identity with v_4, we have

$$u_1 . v_1 v_4 + u_2 . v_2 v_4 + u_3 . v_3 v_4 = 0$$

and therefore forms a_1, a_2, a_3 will exist, such that

(3)
$$\begin{aligned} v_1 v_4 &= a_3 . u_2 - a_2 . u_3 \\ v_2 v_4 &= a_1 . u_3 - a_3 . u_1 \\ v_3 v_4 &= a_2 . u_1 - a_1 . u_2 \end{aligned}$$

a_1, a_2, a_3 will be λ lines and x lines.

Composing the first identity with v_1.

$$0 = v_1 a_3 . u_2 - v_1 a_2 . u_3.$$

(4) Hence $v_1 a_3 = b_1 . u_3$, $v_1 a_2 = b_1 . u_2$, where b_1 is a number. Similarly

$v_2 a_3 = b_2 . u_3$, $v_2 a_1 = b_2 . u_1$, $v_3 a_1 = b_3 . u_1$, $v_3 a_2 = b_3 . u_2$, $v_4 a_1 = b_4 . u_1 \dots.$
where b_2, b_3, b_4 are three other numbers.

Composing the original identity with v_1, we have

$$u_2 . v_1 v_2 + u_3 . v_1 v_3 + u_4 . v_1 v_4 = 0,$$

or
$$u_2 . v_1 v_2 + u_3 . v_1 v_3 + u_4 (a_3 u_2 - a_2 u_3) = 0,$$

consequently a λ and x line a_4 will exist such that

(3)
$$\begin{aligned} v_1 v_2 &= a_4 . u_3 - a_3 . u_4 \\ v_3 v_1 &= a_4 . u_2 - a_2 . u_4 \end{aligned}$$

and similarly
$$v_2 v_3 = a_4 . u_1 - a_1 . u_4 .$$

Composing $v_1 v_2 = a_4 u_3 - a_3 u_4$ with v_3, we have

$$u_4 w = a_4 v_3 . u_3 - a_3 v_3 . u_4$$

or
$$u_4 w = b_3 u_4 . u_3 - a_3 v_3 . u_4$$

and (5)
$$w = b_3 . u_3 - a_3 v_3 .$$

Composing the original identity with a_3

$$\begin{aligned} &a_3 v_1 . u_1 + a_3 v_2 . u_2 + a_3 v_3 . u_3 + a_3 v_4 . u_4 = 0 \\ &b_1 u_3 . u_1 + b_2 u_3 . u_2 + (b_3 u_3 - w) u_3 + b_4 u_3 . u_4 = 0 \end{aligned}$$

$$w = b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 \tag{6}$$

w therefore belongs to the linear system of u_1, u_2, u_3, u_4.

From the identity

$$v_1 v_2 . v_3 l + v_2 v_3 . v_1 l + v_3 v_1 . v_2 l = v_1 v_2 v_3 . l$$

where l is any λ line, inserting the values above found

$$a_4 . (v_1 l . u_1 + v_2 l . u_2 + v_3 l . u_3) - u_4 . (a_1 . v_1 l + a_2 . v_2 l + a_3 . v_3 l) = - u_4 . W . l.$$

But $v_1 l . u_1 + v_2 l . u_2 + v_3 l . u_3 = - v_4 l . u_4$ owing to the original identity. Consequently

$$a_1 . v_1 l + a_2 . v_2 l + a_3 . v_3 l + a_4 . v_4 l = W . l. \tag{7}$$

Composing this with some point M

$$a_1 M . v_1 l + a_2 M . v_2 l + a_3 M . v_3 l + a_4 M . v_4 l = W . l M$$

or

$$a_1 M . v_1 + a_2 M . v_2 + a_3 M . v_3 + a_4 M . v_4 = W . M. \tag{8}$$

Again, identifying l in 7 with a_1 and utilizing (4) and (5)

$$b_1 . a_1 + b_2 . a_2 + b_3 . a_3 + b_4 . a_4 = 0. \tag{9}$$

Hence

$$a_1 . (b_4 . v_1 l - b_1 . v_4 l) + a_2 (b_4 . v_2 l - b_2 . v_4 l) + a_3 (b_4 . v_3 l - b_3 . v_4 l) = W . b_4 l$$

by means of (7) and (9).

But a_1, a_2, a_3 and l considered as λ lines can only be connected by one linear identity. Hence it follows

$$\begin{aligned} a_1/a_2/a_3 &= \varepsilon . b_4 . W \\ a_1/a_2/l &= \varepsilon (b_4 . v_3 l - b_3 v_4 l) \quad \text{or simpler} \end{aligned}$$

$$a_1/a_2 = \varepsilon (b_4 . v_3 - b_3 . v_4) \tag{10}$$

and similarly

$$\begin{aligned} a_2/a_3 &= \varepsilon (b_4 . v_1 - b_1 . v_4) \\ a_3/a_1 &= \varepsilon (b_4 . v_2 - b_2 . v_4) \end{aligned}$$

where ε is a numeric constant not only in the λ plane, but also in the x plane, whose value we shall assume to be, for simplicity, $= 1$. Similar relations obviously hold for $a_1 a_2 a_4$, $a_1 a_4$, $a_2 a_4$ and $a_3 a_4$.

Multiplying the original identity by b_4 and inserting the value of W, we obtain

$$(11)\quad u_1 . (b_4 . v_1 - b_1 . v_4) + u_2 . (b_4 . v_2 - b_2 . v_4) + u_3 . (b_4 . v_3 - b_3 . v_4) + W . v_4 = 0$$

These eleven relations are sufficient and necessary to explain the connections existing between the various forms introduced.

u_1, u_2, u_3, u_4 being given, all the other forms are determined. We may ask how far the giving of some of the forms of the set above mentioned determines the whole set. Let $b_1, b_2, b_3, b_4, a_1, a_2, a_3, a_4$ be given in accordance with (9). W is then determined by (10). Of the set v_1, v_2, v_3, v_4 any one may yet be arbitrarily chosen, but then, on account of (10) the whole set is known. After this, on account of (4) and (5), also u_1, u_2, u_3, u_4 are known. $b_1 . u_2$, for inst., is $v_1 a_2$, $b_1 . u_1 = W + a_1 v_1$, $b_1 . v_4 = b_4 . v_1 - a_2/a_3$. The u_1, u_2, u_3, u_4 so found will be connected by

$$u_1 . v_1 + u_2 . v_2 + u_3 . v_3 + u_4 . v_4 = 0$$

where the v_1, v_2, v_3, v_4 have the above significance, since

$$\begin{aligned} b_1^2 . v_1 v_2 v_3 &= v_1 (b_2 . v_1 - a_3/a_4)(b_3 . v_1 - a_4/a_2) \\ &= v_1 (a_3/a_4)(a_4/a_2) = v_1 a_4 . a_3 a_4 a_2 = b_1 u_4 . b_1 W \end{aligned}$$

and $v_1 v_2 v_3 = u_4 . W$, etc.

It remains now to throw these relations into a geometrical garb and incidentally to state the cross-connections of these forms in a variety of shapes.

If l is a λ line with constant coefficients a_1/a_2 l is a conic. This conic will contain three points independent of the choice of l. The proof of this lies in the identity, L, M, N denoting arbitrary λ points

$$\begin{vmatrix} a_1 L & a_2 L & l L \\ a_1 M & a_2 M & l M \\ a_1 N & a_2 N & l N \end{vmatrix} = a_1 a_2 l . L M N$$

when it is taken into consideration that the three conics

$$\begin{vmatrix} a_1 L & a_1 M & a_1 N \\ a_2 L & a_2 M & a_2 N \end{vmatrix} \text{ have three points in common.}$$

The three points thus defined will be written (a_1/a_2). They are situate on W, because $W \equiv a_1/a_2/a_3$.

The two triplets of points (a_1/a_2) and (a_1/a_3) are coresidual on W. More accurately stated, the two conics

$$a_1/a_2/l \text{ and } a_1/a_3/l$$

intersect W, besides in (a_1/a_2) and (a_1/a_3), in 3 identical points for brevity denoted by (l). This is evident from the identical relation

$$a_1/a_2/l \,.\, a_1/a_3/m - a_1/a_3/l \,.\, a_1/a_2/m = a_1/a_2/a_3 \,.\, a_1/l/m$$

where m is any λ line. For $a_1/a_2/a_3$ is $\equiv W$ and m may be so determined that $a_1/a_2/l$ and $a_1/a_2/m$ intersect in no more than three points on W.

To each λ point L correspond x lines $a_1 L$, $a_2 L$. To a line of λ points corresponds a pencil of x lines. Hence the ∞^2 pairs of lines $a_1 L$, $a_2 L$, where L is variable, represent a linear transformation of the x plane, that may be briefly denoted by (1, 2). If l is a λ line, a_1/l denotes a pencil of x lines, whose vertex may be (a_1/l). (a_1/l) and (a_2/l) are x points corresponding to each other by virtue of (1, 2). For if L and M are points on l, $a_1 L$ and $a_1 M$ intersect in (a_1/l), and $a_2 L$ and $a_2 M$ intersect in (a_2/l).

The triplet (a_1/a_2) is the self corresponding triangle of (1, 2). Indeed if P is such a point that it corresponds to itself by virtue of (1, 2), then a λ line l must exist so that

$$a_1/l \equiv P, \quad a_2/l \equiv P.$$

Hence if we compose in the x plane

$$a_1 P \equiv l \equiv a_2 P$$

and $a_1 P/a_2 P/m = 0$ however the λ line m may be chosen. Consequently P is one of the three points $a_1/(a_1/a_2)$.

The conic $a_1/a_2/l$ contains the two points (a_1/l) and (a_2/l). Denoting by L, M two points on l, we have $LM = l$, and

$$a_1 L \,.\, a_2 M - a_1 M \,.\, a_2 L = a_1/a_2/LM.$$

The conic contains therefore the point of intersection of $a_1 L$ and $a_1 M$. It contains, besides, the point of intersection of $a_1 L$ and $a_2 L$, i. e. of any two lines corresponding by virtue of (1, 2) whose corresponding λ point is situated on l.

Consider now the three triplets (a_1/a_2), (a_1/a_3), (a_2/a_3). Any three conics through them, $a_1/a_2/l$, $a_2/a_3/l$, $a_3/a_1/l$, that intersect W in the same 3 residual points, intersect besides in three points [$a_1/a_2/l$ and $a_1/a_3/l$ in (a_1/l), $a_1/a_2/l$ and $a_2/a_3/l$ in (a_2/l), $a_2/a_3/l$ and $a_1/a_3/l$ in (a_3/l)] which correspond to each other by virtue of the correspondences (1, 2), (1, 3) and (2, 3). This argument and result is in nowise restricted to any special cubics or special coresidual triplets on them. However we select three coresidual triplets on a cubic, the conics through them intersecting the cubic in an identical triplet of points intersect besides in three points which are correspondents in two linear transformations of the plane.

We shall now place ourselves in the viewpoint of neglecting all of the preceding equations except such as refer to the a_1, a_2, a_3, a_4 and W. And we shall give our attention to the study of those properties of these forms as apply to any cubic W. Let W be given and let a linear transformation, or collineation of its plane whose self-corresponding triangle (a_1/a_2) is on W, be arbitrarily selected and called (1, 2). Then any conic through (a_1/a_2) and a corresponding pointpair intersects W in a triplet (l) and conversely any conic through (a_1/a_2) contains just one corresponding pointpair, as easily is shown by elementary considerations. An auxiliary λ plane may then be constructed and forms a_1, a_2 calculated.

Let now a point P be arbitrarily selected on W. Through (a_1/a_2) we construct a conic $a_1/a_2/l$ that intersects W in a residual triplet (l). This conic also contains the point (a_1/l). Through (l), P and (a_1/l) a conic is determined, that we call $a_1/a_3/l$ and which intersects W, besides in (l) and P, in two points which with P form a triplet called (a_1/a_3). If similarly another point Q is arbitrarily chosen on W then a triplet (a_2/a_3) may be similarly determined so as to contain Q. And now the whole set a_1, a_2, a_3 and the transformations (1, 2), (1, 3), (2, 3) are fixed, because conics through (a_1/a_2), (a_1/a_3) and (a_2/a_3) intersecting W in an identical triplet (l) have their fourth point of intersection in corresponding points.

If the two points whose free choice led to the determination of a_3 are varied while everything else remains the same, then only such λ lines a_3' will be generated as are linearly dependent upon a_1, a_2, a_3. Indeed $a_1/a_2/a_3$ will be, according to the above construction, $\equiv W$, the cubic under discussion. Also $a_1/a_2/a_3'$ will be $\equiv W$. Constants b and b' must therefore exist so that $a_1/a_2/(b\,a_3 - b'\,a_3') = 0$. Let $b\,a_3 - b'\,a_3' = \gamma$. From $a_1/a_2/\gamma = 0$ it follows, if L, M, N are λ points

$$\begin{vmatrix} a_1 L & a_1 M & a_1 N \\ a_2 L & a_2 M & a_2 N \\ \gamma L & \gamma M & \gamma N \end{vmatrix} = 0.$$

But $$\begin{vmatrix} a_1 L & a_1 M \\ a_2 L & a_2 M \end{vmatrix}, \quad \begin{vmatrix} a_1 M & a_1 N \\ a_2 M & a_2 N \end{vmatrix}, \quad \begin{vmatrix} a_1 L & a_1 N \\ a_2 L & a_2 N \end{vmatrix}$$

have three points in common. Hence γM must contain the 4th point common to the two first determinant conics and, γ being a straight line, it may easily be shown that γ must be a linear combination, with constant coefficients, of a_1 and a_2.

The relation between the 6 triplets on $W(a_1/a_2) \ldots (a_3/a_4)$ is then that they are coresidual and that the 6 conics through these 6 triplets and any residual triplet on W intersect in only 4 more points. We have shown above that this statement necessarily involves the linear dependence of a_1, a_2, a_3, a_4 expressed by (9). Equation (9) may find expression in a geometrical shape as follows: L being any λ point, $a_1 L$, $a_2 L$, $a_3 L$, $a_4 L$ therefore x lines, from

$$b_1 . a_1 L + b_2 . a_2 L + b_3 . a_3 L + b_4 . a_4 L = 0$$

it is evident that $b_1 . a_1 L + b_2 . a_2 L$ is the line passing through the intersections of $a_1 L / a_2 L$ and through that of $a_3 L / a_4 L$. Now to each L corresponds a definite $b_1 . a_1 L + b_2 . a_2 L$, and $b_1 . a_1 + b_2 . a_2$ is therefore in conjunction with a_1 a symbol for a definite linear transformation of the x plane, whose self-corresponding triangle must be on W. In fact this triangle is no other than the three points common to $a_1 / b_1 a_1 + b_2 a_2 / l$, i. e. no other than the triplet (a_1/a_2). Hence by some linear transformation T whose self-corresponding triangle is (a_1/a_2), each $a_1 L$ is converted into the line joining the intersections of $a_1 L / a_2 L$ and $a_3 L / a_4 L$. And a similar statement holds for the other diagonals of the quadrangle of corresponding lines $a_1 L$, $a_2 L$, $a_3 L$, $a_4 L$.

We shall now consider the forms u_i and v_i that were hitherto neglected. If L is any λ point, $v_1 v_2 L$, $v_2 v_3 L$, $v_3 v_1 L$ are three quartics that have 12 points in common; for it is clear that $v_1 v_2 L$ and $v_2 v_3 L$ contain the 4 points of the pencil $v_2 L$ and have their other common points on $v_1 v_3 L$. From the identity

$$v_1 v_2 v_3 . L = v_1 v_2 L . v_3 + v_2 v_3 L . v_1 + v_3 v_1 L v_2$$

it follows, since $v_1 v_2 v_3 = u_4 . W$, that these 12 points common to $v_1 v_2 L$, $v_2 v_3 L$, $v_3 v_1 L$ either lie on W or on u_4. Now $v_1 v_2 = a_4 u_3 - a_3 u_4$, hence $v_1 v_2 L$ intersects u_4 in the 9 points, where u_3 cuts that curve and besides in the 3 points of intersection of $a_4 L$ and u_4. Consequently $v_1 v_2 L$, $v_2 v_3 L$, $v_3 v_1 L$ have three of their common points on u_4, nine on W. $v_1 v_2 \equiv v_1 (b_2 v_1 - b_1 v_2) \equiv v_1 (a_3/a_4)$ intersects W always in the 3 points (a_3/a_4). Consequently the 9 variable points of the intersection of $v_1 v_2 L$ and W are those that are common to $v_1 v_3 L$ and $v_2 v_3 L$; and, by the same reasoning, also to $v_1 v_4 L$, $v_2 v_4 L$, $v_3 v_4 L$.

$v_1 v_2 L$ and $v_3 v_4 L$ have, beside these 9 points, still 7 points in common.

These complete with the 18 points (u_1, u_2), (u_3, u_4) the base of 25 points of a pencil of quintics. Indeed the two quintics.

$$u_1 \,.\, v_1 L M + u_2 \,.\, v_2 L M$$

and

$$u_1 \,.\, v_1 L N + u_2 \,.\, v_2 L N$$

contain the 9 points (u_1, u_2), also those of (u_3, u_4) owing to identity (1), and besides 7 points obviously situate on the result of eliminating u_1, u_2 from above,

$$\begin{vmatrix} v_1 L M & v_2 L M \\ v_1 L N & v_2 L N \end{vmatrix} = v_1 v_2 L \,.\, L M N.$$

By the same reasoning and by means of the equation (1) these 7 points are also situate on $v_3 v_4 L$. Moreover, it is clear that none of these points will generally lie on say $v_1 v_3 L$. The group of 7 points is therefore characterized as that part of the intersection of $v_1 v_2 L$ and $v_3 v_4 L$ that is residual to the group of 9 points common to the $v_i v_j L$.

The 7 points $(v_1 v_2 L, v_3 v_4 L)$ form with the point of intersection of $a_1 L$, $a_2 L$ and with that of $a_3 L$, $a_4 L$ the base of a pencil of cubics. Since $u_1 \,.\, v_1 L M + u_2 \,.\, v_2 L M$ as well as $v_3 v_4 L = a_2 L \,.\, u_1 - a_1 L \,.\, u_2$ contain the 7 points, so will $a_1 L \,.\, v_1 L M + a_2 L \,.\, v_2 L M$. Hence this cubic and $a_1 L \,.\, v_1 L N + a_2 L \,.\, v_2 L N$ contain the 7 points. The two cubics contain, besides, point $(a_1 L, a_2 L)$, and, owing to relations (7) and (8), also $(a_3 L, a_4 L)$.

From all this it is then apparent: Let S be a linear system of cubics which has 4 linearly independent cubics (u_1, u_2, u_3, u_4) as base and which, besides, is not of that particular nature that the 18 points (u_1, u_2), (u_3, u_4) should admit more than 3 linearly independent quintics through them. Then two quintics through two groups of nine points common to two cubics of S, for inst. (u_1, u_2) and (u_3, u_4), will intersect in 7 residual points that with each one of the above group of nine points will lie on *one* quartic. The two quartics thus determined will have 9 residual points in common, and these 9 points will always lie on the same cubic W. The cubic thus determined belongs to the system S. Moreover, it is clear from what precedes, that however the selection of the original two pairs of cubics in S and of the two quintics through the pair of nine points may be varied the group of nine points determining W must always be one of a determinate system Γ of ∞^2 such nine-point groups, as any particular nine-point group of Γ does not depend at all on the choice of the two pairs of cubics (variety of choice obviously subjecting the v_1, v_2, v_3, v_4 only to a linear transformation of each other) but entirely on the choice of the two quintics through the 18 points, i. e. on the λ line l.

The cubic W of the system S has still other remarkable properties easily derived from the set of equations previously deduced. Thus (11) shows that each quintic

$$u_1 (b_4 . v_1 l - b_1 v_4 l) + W . v_4 l$$

also contains the 9 points (u_2, u_3). But $b_4 . v_1 l - b_1 . v_4 l$ has, wherever l may be situated, the 3 points (a_2/a_3) in common with W. Hence, a cubic (u_1) of S intersects W in 9 points, that with any nine points common to two other cubics of S (u_2, u_3) determine a triplet of points (a_2/a_3) such that any quintic through above 18 points also contains the triplet. And this triplet is situate on W, and all triplets thus determined, by varied choice of the cubics (u_1, u_2, u_3) in S, are coresidual. Moreover, each such triplet is residual to any one of the nine-point groups of Γ, and a quartic $(v_1 v_4 l)$ containing any one such triplet and an individual of Γ contains also nine points of intersection of two cubics of S (u_2, u_3).

A group of nine points common to two cubics of S is determined when two of its points are arbitrarily given, for a cubic of S may be made to pass through any three points, and two points determine therefore a pencil in S. The ∞^4 system Δ of nine points forming the base of a pencil in S, and the system Γ of nine point groups has this relationship that any two individuals of Δ and Γ lie on a quartic. Let for inst. a nine point group of Δ be the intersection of u_1', u_2'. Choosing as base of S u_1', u_2', u_3', u_4' (any 4 linearly independent cubics two of which are u_1', u_2') it is clear from the preceding, that the corresponding v_i' are such as to yield quartics $v_3' v_4' L$ containing, according to the selection of L, any given individual of Γ and (u_1', u_2').

Not every quartic through an individual of Γ contains an individual of Δ. Let a given individual Γ_L of Γ be common to $v_1 v_2 L$, $v_1 v_3 L \dots . v_3 v_4 L$. Then any quartic through Γ_L will be

$$f = \xi_{1,2}\, v_1 v_2 L + \dots . + \xi_{3,4}\, v_3 v_4 L$$

where $\xi_{1,2} \dots . \xi_{3,4}$ are 6 indeterminates. Should f contain an individual of Δ it must be representable in the shape $v_i' v_j' L$, where v_i' and v_j' are linearly dependent upon the v_i. This, as in the case of straight lines in space, requires an equation to be satisfied, namely $\xi_{1,2} \cdot \xi_{3,4} + \xi_{2,3} \cdot \xi_{1,4} + \xi_{3,1} \cdot \xi_{2,4} = 0$.

The same method leads to interesting results when 4 curves of order n are under discussion. And it is a suggestive fact that the connections between 4 forms of order $(n-1)$, $(n-2)$, etc., recur again at the higher orders. In the case $n=3$ that has just been studied, for instance, the equation

$$a_1 M . v_1 l + a_2 M . v_2 l + a_3 M . v_3 l + a_4 M . v_4 l = 0$$

where M is a variable point on l, shows that the a_1/l, a_2/l, a_3/l, a_4/l are pencils of lines that stand to the conics $v_1 l$, $v_2 l$, $v_3 l$, $v_4 l$ in the same report as the pencils previously named g_1, g_2, g_3, g_4 to the 4 conics of the case $n=2$. And the analogy goes very much further. So is the triangle EFG in the case $n=3$ represented by the triangle (l) on W, and all the other points and lines that were mentioned in the configuration belonging to $n=2$ find ready interpretation for the set of conics $v_2 l$.

If the condition referring to the 18 points (u_1, u_2), (u_3, u_4) is not satisfied, more than three linearly independent quintics will exist containing the 18 points and the v_1, v_2, v_3, v_4 will then be points in a λ space of manifoldness 4. It follows then

$$v_1 v_2 v_3 = u_4 . W, \text{ etc.},$$

where W is a λ plane containing v_1, v_2, v_3, v_4 and is of order 3 in the x coordinates or else vanishes. Again we have the equations

$$v_1 v_2 = a_4 u_3 - a_3 u_4, \text{ etc.},$$

where the a_i are of 1st order in the x_i, and of the dimension of a line in the λ space. Obviously

$$(a_4 . u_3 - a_3 . u_4)(a_4 . u_3 - a_3 . u_4) = 0$$

i. e. $a_4 . u_3 - a_3 . u_4$ composed with itself is 0. Hence

$$a_4 a_4 . u_3^2 - 2 a_4 a_3 . u_4 u_3 + a_3 a_3 . u_4^2 = 0.$$

Consequently $a_4 a_4$ is divisibly by u_4. But $a_4 a_4$ is only of order 2 in the x_i. Therefore

$$a_4 a_4 = 0, \; a_3 a_3 = 0, \; a_4 a_3 = 0.$$

Considered as λ forms, $v_1 v_2$, $a_4 a_3$ are three lines having a common point of intersection. The same argument holds in respect to v_1, v_3 and a_2, a_4, etc. The v_1, v_2, v_3, v_4 must therefore be λ points in a λ plane W in which also a_1, a_2, a_3, a_4 are situated.

From $v_1 v_2 = a_4 u_3 - a_3 u_4$ it follows that $v_1 a_4$ is divisible by u_4. Let be

$$v_1 a_4 = b_1 . u_4$$

b_1 will then be (unless it vanishes) a λ space independent of the x plane. We shall also have $v_1 a_3 = b_1 . u_3$ and generally $v_i a_j = b_i . u_j$ if $i \neq j$.

To find $v_i a_i$, we compose

$$v_1 v_2 = a_4 u_3 - a_3 u_4$$

with v_3.
$$u_4 . W = b_3 u_4 . u_3 - v_3 a_3 u_4$$

or
$$v_3 a_3 = b_3 . u_3 - W.$$

Composing $u_1 v_1 . + \dots + u_4 . v_4$ with a_1 we obtain

$$b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 = W.$$

$b_1 v_1$ is $= 0$. Consequently

$$b_2 v_1 . u_2 + b_3 v_1 . u_3 + b_4 v_1 . u_4 = 0.$$

This shows $b_2 v_1 = 0$, $b_3 v_1 = 0$, $b_4 v_1 = 0$. The b_i *must* vanish, and therefore also W. The v_i will all be on the same λ line a and the a_i will be, as λ lines, congruent with a. From this $a_i = c_i . a$, where c_i is a numeric. Therefore

$$v_1 v_2 = (c_4 u_3 - c_3 u_4) . a, \text{ etc.,}$$
$$v_1 (c_2 v_2 + c_3 v_3 + c_4 v_4) = 0$$

and
$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0.$$

Let $a = x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3$, where x_1, x_2, x_3 are the x_i variables, α_1, α_2, α_3 lines in the λ space independent of x. Now $(v_1 v_2)(v_1 v_2) = 0$. Hence $aa = 0$, and it follows $\alpha_i \alpha_j = 0$. The α_i are intersecting lines. They cannot lie in the same λ plane P, or $P v_i = 0$, and the v_1, v_2, v_3, v_4 would be forms of a plane, not of a space of 4 manifoldness, contrary to hypothesis. Consequently the α_i must have a λ point G in common.

Let then be $a = \alpha G$, where α is a λ point. $v_i G$ is $\equiv a$, therefore let be

$$v_i G = p_i . a$$

where p_i is a linear x form. Then

$$v_i G = p_i . \alpha G$$

and
$$v_i = p_i . \alpha + q_i . G.$$

The original identity, after the insertion of these values, gives then the two equations

$$p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4 = 0$$
$$q_1 u_1 + q_2 u_2 + q_3 u_3 + q_4 u_4 = 0$$

where the p_i are of the 1st order, the q_i of the 2nd order. From this, as before,

$$p_1 q_2 - p_2 q_1 = c_4 u_3 - c_3 u_4\,,\ \text{etc.},$$
$$c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 = 0$$
$$c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 = 0$$

Also

$$(c_1 u_2 - c_2 u_1) p_2 + (c_1 u_3 - c_3 u_1) p_3 + (c_1 u_4 - c_4 u_1) p_4 = 0$$
$$(c_1 u_2 - c_2 u_1)\, q_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

This can only be, if the three cubics $c_1 u_2 - c_2 u_1$, $c_1 u_3 - c_3 u_1$, $c_1 u_4 - c_4 u_1$ have 7 points in common.

The salient fact then is, that whenever the 18 points (u_1, u_2), (u_3, u_4) are such as to permit more than three linearly independent quintics through them, some identity $p_1 u_1 + p_2 u_2 = -p_3 u_3 - p_4 u_4$ will exist and therefore a quartic will contain all the points. When the above condition is satisfied, a quintic form f will exist apolar to u_1, u_2, u_3, u_4. And conversely *any* quintic f being given, 4 cubics u_1, u_2, u_3, u_4 may be found that are apolar to f. The 7 points common to $c_1 u_2 - c_2 u_1$, etc., may be denoted by $A \ldots . G$. These 7 points will be those by means of which f may be represented as sum of seven fifth powers of linear forms $A^5 + \ldots . + G^5$. We need not dwell on this point that is easily made plain but by considerations foreign to the subject of this paper.

Now much of what has been said and done here may be generalized. A form of order $2n-1$ being arbitrarily given, the set of forms of order n apolar to it may be studied in an analogous fashion. Nor is this calculation restricted in any way in the number of variables x_i.

The calculation may be generalized in a somewhat different direction and yields, without much labor, an interesting and very applicable result. As a starting point we use the theory of my paper "Zur Theorie der Moduln und Ideale." Accordingly the reader will be supposed to understand the meaning of the form $\Omega\,(u_1 \ldots . u_m)$ and to be acquainted with propositions I, II, III of that paper.

Let $u_1 \ldots . u_m$ be m forms of m homogenous variables, whose resultant does not vanish. Their orders may be designated by $n_1 \ldots . n_m$. A form of order

$n_1 + \dots + n_m - m$ will then exist apolar to all of them. This form may be written Ω. There will be no form of order $n_1 + \dots + n_m - m$ other than Ω apolar to $u_1 \dots u_m$, and no form of higher order will be apolar to the set (Proposition III). Besides Ω will have this remarkable property, not hitherto laid down, that any form of lower order than Ω apolar to $u_1 \dots u_m$ must necessarily be a polar form of Ω. With other words, if ψ is a form apolar to $u_1 \dots u_m$, of order $n_1 + \dots + n_m - m - N$, where N is a positive integer, then a form g of order N will exist such that identically

$$\psi = g \times \Omega$$

i. e. that ψ is the polar of g in respect to Ω.

To show this, let F be a form of order $n_1 + \dots + n_m - m - N$, f one of order N, and let all the coefficients of F and f be indeterminate quantities. The magnitude $F . f \times \Omega$ is then a bilinear form F of the indeterminate coefficients of F and f. If it is required that $F . f \times \Omega = 0$ while the set of coefficients of F remains arbitrary, then the set of coefficients of f will be subject to a certain number α of conditions which remains the same, if F and f change their roles in this. This is merely an expression of one of the elementary properties of bilinear forms.

It follows then that the number of conditions imposed upon a form of order $n_1 + \dots + n_m - m - N$, to be apolar to Ω is the same as the corresponding number for forms of order N. And the number of contragredient forms of order $n_1 + \dots + n_m - m - N$ apolar to $u_1 \dots u_m$ has the same value. Let this number be denoted by α.

Exactly α forms $g_1 \dots g_\alpha$ of order N will then exist, which are linearly independent and no linear combination of which belongs to the module $(u_1 \dots u_m)$, or, what is the same, is apolar to Ω. Let g represent an indeterminate form of this linear system. Then $g \times \Omega$ will never vanish, and will contain α indeterminatae. But the linear system of forms of order $n_1 + \dots + n_m - m - N$ apolar to $u_1 \dots u_m$ has the manifoldness α, and there is only one such system if two forms congruent modulo $(u_1 \dots u_m)$ are for this purpose considered equivalent or identical. Consequently the set $g \times \Omega$ represents that system; and if ψ is a form of order $n_1 + \dots + n_m - m - N$ apolar to $u_1 \dots u_m$ then a form g will indeed exist so that $\psi = g \times \Omega$.

This theorem immediately leads to any number of geometrical propositions. Let us apply it, for inst., to the proposition concerning the 18 points (u_1, u_2),

(u_3, u_4) through which more than 3 linearly independent quintics pass. A quintic ψ will then exist apolar to u_1, u_2, u_3, u_4. The Ω of u_1, u_2, u_3 is of order $3+3+3-3=6$. ψ being a quintic apolar to u_1, u_2, u_3 a linear form g exists such that $\psi = g \times \Omega$. But $u_4 \times \psi = 0$, consequently $u_4 \cdot g \times \Omega = 0$, and $u_4 \cdot g$ belongs to the modulus u_1, u_2, u_3. This shows at once that the 18 points (u_1, u_2), (u_3, u_4) are on a quartic.

If as a particular case 16 of the 18 points are the base of a pencil of quartics — when indeed the supposition will be satisfied — then the present proposition shows itself to be identical with the one announced previously.

It would naturally not at all be difficult to draw similar conclusions for curves of higher orders or for forms of higher manifoldness.

In this point the proposed method of calculating geometric configurations and the general theory of moduli meet. In all probability the connection between the two disciplines will grow much more intimate as the method will further develop. It is clear that equations of the type $\Sigma u_i \cdot v_i = 0$ with their λ space adjoined are very apt to express the properties of systems of forms of certain orders which belong to two different modules simultaneously, which, for inst., contain two distinct irreducible or reducible geometric configurations. And from the preceding examples it is fairly evident that the proposed treatment of such equations yields results.

It is true that only a few and perhaps rather simple examples have been discussed in what precedes. But the author may be excused if he points out that it would have been easy enough to extend the method much further and that the difficulty for him consisted rather in limiting the examples to such as would bring out, in a lucid and easy manner, some of the characteristic properties of the proposed method.

NEW YORK, *May* 3, 1906.

Groups Generated by n Operators Each of Which is the Product of the $n-1$ Remaining Ones.

By G. A. Miller.

The case when $n=3$ has recently been considered.* When $n=2$ the groups are evidently cyclic and hence require no consideration in this connection. In the present paper we shall consider $n>3$, and we shall first assume that the products of the $n-1$ operators are independent of their orders and hence all of them must be commutative. Representing the n operators under consideration by $s_1, s_2, \ldots, s_n$ we have by hypothesis, s_n being any one of the n operators, that

$$s_1s_2\ldots s_{n-1}=s_n, \text{ or } s_1s_2\ldots s_n^{-1}=1.$$

From the two equations

$$s_1s_2\ldots s_{n-1}=s_n \text{ and } s_1s_2\ldots s_{n-2}s_n=s_{n-1}$$

it follows, by multiplying one into the inverse of the other, that *any two of these n operators have the same square* and, by direct multiplication, that *the* $2(n-2)^{\text{th}}$ *power of each operator is the identity.*

If we substitute for $s_1, s_2, \ldots, s_{n-1}$ the $n-1$ independent transpositions $a_1b_1, a_2b_2, \ldots, a_{n-1}b_{n-1}$, there results a system of operators which satisfy the given conditions for every value of $n>3$. These n operators clearly generate the Abelian group of order 2^{n-1} and of type $(1, 1, 1, \ldots)$. From the given theorem it results that this is the only system of Abelian groups of type $(1, 1, 1, \ldots)$ which may be generated by n operators satisfying the given condition, if we exclude the trivial case when the group is cyclic. By letting $s_1=s_2=\ldots=s_n$ it is clear that the cyclic group of order $n-2$ might be said to be generated by operators satisfying the given condition. To avoid the consideration of such trivial cases we shall assume that no two of the n operators under consideration are identical. From this assumption and the given theorem it follows that no more than one of them can be of odd order, and if the order of one of them is an odd number the order of the others is twice this odd number.

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*Bulletin of the American Mathematical Society, vol. 13 (1907), p. 381.

Hence the theorem: *If the order of one of the n operators* $s_1, s_2, \ldots, s_n$ *is divisible by* 4, *all of them have the same order. If this condition is not satisfied, either all of them have for their order the double of the same odd number or* $n-1$ *of them have this order while the remaining one has the odd number for its order.*

As the operators $s_1, s_2, \ldots, s_n$ have a common square and are commutative, we have the equations $s_\alpha^2 = s_\beta^2$, $s_\alpha s_\beta^{-1} = s_\alpha^{-1} s_\beta$, $(s_\alpha s_\beta^{-1})^2 = s_\alpha s_\beta^{-1} s_\alpha s_\beta s_\beta^{-2} = 1$. That is, each of these operators may be obtained by multiplying any other one by some operator of order 2. Hence all of them may be obtained by multiplying one by the identity and different operators of order 2. On the other hand, it may be observed that if $t_1^2 = t_2^2$ and if $t_1 = \rho t_2$, where $\rho^2 = 1$, it is necessary that $t_1 t_2 = t_2 t_1$. For, as the second equation near the beginning of this paragraph does not imply that $s_\alpha s_\beta = s_\beta s_\alpha$ it follows that $t_1 t_2^{-1} = t_1^{-1} t_2$, or $\rho = t_1^{-1} t_2$. Hence $t_1^{-1} t_2 t_1^{-1} t_2^{-1} t_2^2 = t_1^{-1} t_2 t_1^{-1} t_2^{-1} t_1^2 = t_1 t_2 t_1^{-1} t_2^{-1} = 1$. From this it results that the commutator of t_1, t_2 is the identity and hence these operators are commutative. We have thus arrived at the theorem: *The necessary and sufficient condition that two different operators which have a common square are commutative is that one is the product of the other into an operator of order* 2.

From the preceding paragraph it follows that the n operators under consideration may be represented as follows: $s_1, \rho_1 s_1, \rho_2 s_1, \ldots, \rho_{n-1} s_1$; where $\rho_1, \rho_2, \ldots, \rho_{n-1}$ represent $n-1$ different operators of order 2 which are commutative with each other and with s_1. Since

$$s_1 = \rho_1 s_1 \cdot \rho_2 s_1 \cdot \ldots \cdot \rho_{n-1} s_1 = \rho_1 \rho_2 \cdot \ldots \cdot \rho_{n-1} \cdot s_1^{n-1}$$

and $s_1^{2(n-2)} = 1$, it results that $\rho_1 \rho_2 \cdot \ldots \cdot \rho_{n-1} = s_1^{n-2}$. The n commutative operators $s_1^{n-2}, \rho_1, \rho_2, \ldots, \rho_{n-1}$ must therefore have the property that each of them is equal to the product of all the others. When $s_1^{n-2} = 1$ the $n-1$ operators $\rho_1, \rho_2, \ldots \rho_{n-1}$ have the same property. As the group generated by $s_1, s_2, \ldots, s_n$ is identical with the one generated by $s_1, \rho_1, \rho_2, \ldots, \rho_{n-1}$, we have the interesting theorem: *If a group G is generated by n commutative operators such that each is the product of all the others, then G is the direct product of a cyclic group whose order divides* $2(n-2)$ *and an Abelian group of order* 2^α *and of type* $(1, 1, 1, \ldots)$. Moreover, any such direct product may be generated by n operators which satisfy the given condition.

While $n-2$ of the operators $\rho_1, \rho_2, \ldots, \rho_{n-1}$ can always be replaced by independent transpositions, as was observed in the second paragraph, it may be possible to replace them by operators which generate a much smaller group. For instance, when $n = 2^\beta$ and $s_1^{n-2} = 1$, it is possible to replace all of them by the operators of order 2 in the Abelian group of order 2^β and of type $(1, 1, 1, \ldots)$. If $s_1^{n-2} \neq 1$ and $n = 2^\beta - 1$, they may be replaced by $n - 1$ of the operators of the same Abelian group, while s_1 may be so chosen that its $(n-2)$th power is equal to the remaining operator of order 2. In each of these cases the order of G is either 2^β or $2^{\beta-1}$ into the order of s_1.

Non-Abelian Groups.

When the n operators $s_1, s_2, \ldots, s_n$ are not supposed to be commutative, it is generally possible to select them in such a way as to satisfy the condition expressed in the heading of this article and to generate any one of a large number of different types of groups. This is especially true when $n > 4$, as will appear in what follows. It is, however, possible to establish a few general theorems of interest, and to exhibit many fundamental properties of the possible groups when $n = 4$, by means of elementary considerations. One of these theorems may be stated as follows: *If the n operators* $s_1, s_2, \ldots, s_n$ *are arranged cyclically and the product of any* $n - 1$, *in order, is equal to the remaining one, then all of them have a common square.*

The proof of this theorem follows almost directly from the defining equations; for the two equations

$$s_1 s_2 \ldots s_{n-1} = s_n, \quad s_2 s_3 \ldots s_n = s_1$$

imply $s_1^{-1} s_n = s_1 s_n^{-1}$ and hence $s_n^2 = s_1^2$. Similarly we may prove that $s_1^2 = s_2^2$, etc. Moreover, it results that

$$s_{n-1} s_{n-2} \ldots s_2 s_1 = (s_1 s_2 \ldots s_{n-1})^{-1} \cdot s_1^{2(n-1)} = s_1^{2(n-2)} s_n,$$

and this includes a second proof of the fact that the $2(n-2)$th power of each operator is the identity whenever the n operators are commutative.

If $s_1, s_2, \ldots, s_n$ are any n different operators of order 2 which satisfy the condition

$$s_1 s_2 \ldots s_n = 1 \tag{A}$$

it follows that $s_{\alpha+1} \ldots s_n s_1 s_2 \ldots s_{\alpha-1} = s_\alpha$; $\alpha = 1, 2, \ldots, n$. That is, the product of any $n - 1$ of them in order is the remaining one. Of $n \geqq 5$ the operators of (A) may be so chosen as to generate any symmetric group whose degree exceeds a given number $(m - 1)$. To prove this statement it is only necessary to observe that s_1, s_2 may be so selected as to generate the dihedral group of order $2p$, $m \geqq p > \frac{m}{2}$ and p being prime, according to the well-known theorem due to Tchébycheff. Hence it is possible to choose the three operators (s_1, s_2, s_3)

of order 2 so that they generate a transitive group of degree m involving negative substitutions. This must be the symmetric group, since it involves the cycle of order p and such a cycle cannot occur in any non-symmetric and non-alternating primitive group unless its degree is p, $p+1$, or $p+2$.* If m had one of the last three values it would be easy to select s_1, s_2, s_3 so that the primitive group generated by them would involve a transposition. This completes the proof of the statement in question, since it is only necessary to find an operator of order 2 which transforms $s_1 s_2 s_3$ into its inverse in order to find the five operators of order 2 such that $s_1 s_2 s_3 s_4 s_5 = 1$.

From the preceding paragraph it is clear that the number of different types of groups that may be generated by $s_1, s_2, \ldots, s_n$ $(n > 4)$ is so large as to make it questionable whether it is desirable to endeavor to give an enumeration of all the possible types. When $n = 4$ the matter becomes comparatively simple, and hence we restrict ourselves to this case in what follows. From the equations

$$s_1 s_2 s_3 = s_4, \quad s_2 s_3 s_4 = s_1, \quad s_3 s_4 s_1 = s_2, \quad s_4 s_1 s_2 = s_3$$

we obtain

$$s_1 s_2 s_3 s_4^{-1} = s_1^{-1} s_2 s_3 s_4 = s_1 s_2^{-1} s_3 s_4 = s_1 s_2 s_3^{-1} s_4 = 1.$$

Since s_3, s_4 transform $s_3 s_4^{-1}$ into its inverse,† they must also transform $s_1 s_2$ into its inverse. That is, the product of any two of these operators, taken in cyclical order, is transformed into its inverse by each of the other two. We shall now consider the group (H) generated by the two operators

$$s_1 s_2^{-1}, \quad s_2 s_3^{-1}.$$

Each of these operators is transformed into its inverse by s_2, and s_3^{-1} transforms $s_2 s_1^{-1} = s_2^{-1} s_1^{-1} . s_2^2$ into $s_1 s_2 . s_2^2 = s_1 s_2^{-1} . s_2^4$. That is, $s_2 s_3^{-1}$ transforms $s_1 s_2^{-1}$ into $s_1 s_2^{-1} . s_2^4$. Since s_2^4 is invariant, it follows that $\{s_1 s_2^{-1}, s_2 s_3^{-1}\}$ is metabelian and its commutator subgroup is the cyclic group generated by s_2^4. When the common order of s_1, s_2, s_3, s_4 is either 2 or 4, $\{s_1 s_2^{-1}, s_2 s_3^{-1}\} = H$ is Abelian and the group G generated by s_1, s_2, s_3, s_4 may be obtained by extending H by means of an operator of order 2 or 4 which transforms each operator of H into its inverse. In this case H is either cyclic or the direct product of two cyclic groups.

When H is cyclic G may be any dihedral group whose order exceeds 6, since any such group is generated by four operators of order 2 which satisfy the condition (A). In fact, the two remaining dihedral groups can be generated by four operators satisfying (A) if it is not implied that all the operators are distinct and that none of them is the identity. Hence the theorem: *Every dihedral group may be generated by four operators, each of which is a product of the other three.* When the order of this dihedral group exceeds 6, it may be assumed that the four operators are distinct. By dimidiating* any two dihedral groups with respect to the cyclic subgroups of half their orders we obtain a group G which may be generated by four operators of order 2, each of which is a product of the other three. If s_1', s_2' and s_1'', s_2'' respectively are generators of the dihedral groups in question, each of these operators being of order 2, the four generators of G $s_1' s_1''$, $s_1' s_2''$, $s_2' s_2''$, $s_2' s_1''$ clearly satisfy the conditions imposed on s_1, s_2, s_3, s_4. Hence it follows that *every group which may be obtained by extending the direct product of two cyclic groups by means of an operator of order 2 which transforms each operator of this direct product into its inverse may be generated by four operators of order 2, each of which is a product of the other three.*

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* Bulletin of the American Mathematical Society, vol. 4 (1898), p. 140.

† Archiv der Mathematik und Physik, vol. 9 (1905), p. 7.

When H is an Abelian group of even order, it is well known that we can construct a group G of twice the order of H by adding operators of order 4 which transform each operator of H into its inverse and have a common square. If H is cyclic and not of order 2, it is easy to find four such operators, each of which is a product of the other three. The smallest of these groups is the quaternion, and the four operators j, k, $-j$, $-k$ clearly satisfy the conditions

$$j \, . \, k \, . -j = -k, \quad k \, . -j \, . -k = j, \quad -j \, . -k \, . \, j = k, \quad -k \, . \, j \, . \, k = -j.$$

When H is the direct product of two cyclic groups, G may be constructed by dimidiation just as in the preceding paragraph; and if s_1', s_2' and s_1'', s_2'' are the generators of order 4 of the constituent groups, G may clearly be generated by $s_1' s_1''$, $s_1' s_2''$, $s_2' s_2^{-1''}$, $s_2^{-1'} s_1''$ and these satisfy the condition that each is the product of the other three in cyclic order. The results of this and the preceding paragraph exhaust the possible groups when H is Abelian and includes s_1^2. That is, if the order of s_1 is 2 or 4 and if s_1, s_2, s_3, s_4 are such that the product of any three, in a given cyclic order, is the fourth, then they generate one of the groups considered in this and the preceding paragraph whenever s_1^2 is in H. If s_1^2 is not in the Abelian H, s_1 is necessarily of order 4 and it is necessary to extend H by means of an operator (s_1^2) of order 2 which is commutative with all its operators. The remaining operators of G transform each operator of this extended H into its inverse and have a common square. Moreover, every such extended H will give rise to one G which is generated by four operators of order 4 satisfying the conditions imposed on s_1, s_2, s_3, s_4. Hence when H is Abelian G may be obtained by extending an Abelian group which has at most three invariants (if its maximal invariants are chosen) by means of an operator which transforms each operator of this Abelian group into its inverse.

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* Cayley, Quarterly Journal of Mathematics, vol. 25 (1890), p. 71.

It remains to consider the groups when H is non-Abelian. It has been proved that such an H is metabelian, contains a cyclic commutator subgroup, is invariant under G, and that the order of G is either twice or four times that of H. Moreover, the two generators of H $(s_1 s_2^{-1}, s_2 s_3^{-1})$ are independent of the commutator subgroup of H. That is, neither of these operators generates any commutator besides the identity, since such commutators are generated by s_1^4, and s_1^2 is invariant under G while s_2 transforms both $s_1 s_2^{-1}$ and $s_2 s_3^{-1}$ into their inverses. The orders of $s_1 s_2^{-1}$ and $s_2 s_3^{-1}$ are divisible by the order of s_1^4, and each of the operators s_1, s_2, s_3, s_4 is of even order. The last statement follows from the fact that if s_2 were of odd order it would be commutative with s_2, s_3, s_4, since they have the same square. Hence it would also be commutative with $s_1 s_2^{-1}$, $s_2 s_3^{-1}$ and the orders of these operators could not exceed 2. These operators would therefore be commutative, since s_1^4 could not be of order 2. This proves the theorem: *If the n operators $s_1, s_2, \ldots, s_n$ are arranged cyclically and the product of any $n-1$, in order, is the remaining one, then all are of even order when $n < 5$.*

From the preceding paragraph it follows that H may be constructed by extending the direct product of two cyclic groups, which are such that the order of the one is a divisor of the order of the other, by means of an operator which is commutative with the generator of one of these groups and transforms the generator of the other into the product of the two generators. It follows that the order of the extending operator is also divisible by the order of the invariant generator. Moreover, any such group can be used for H, since the two generating operators in question may be replaced by s_1^4 and $s_1 s_2^{-1}$, and the extending operator may be replaced by $s_2 s_3^{-1}$. It is then possible to find an operator which has the properties imposed on s_2, since it is possible to establish a simple isomorphism of H with itself in which s_1^4 corresponds to itself and each of the operators $s_1 s_2^{-1}$, $s_2 s_3^{-1}$ corresponds to its inverse. The last statement follows from the fact that $s_3 s_2^{-1}$ transforms $s_2 s_1^{-1}$ into $s_1^4 \cdot s_2 s_1^{-1}$ and $s_2 s_3^{-1}$ transforms $s_1 s_2^{-1}$ into $s_1^4 \cdot s_1 s_2^{-1}$. As the quotient group G/H is cyclic and of order 2 or 4, it is easy to construct all the possible G's for any particular H. It may be observed that the properties of all of these groups are somewhat similar to those of the dihedral type. In particular, all of them are solvable.

Concerning Systems of Conics Lying on Cubic Quartic and Quintic Surfaces.

By C. H. Sisam.

INTRODUCTION.

Although the properties of algebraic ruled surfaces have been extensively studied, and the classification of such surfaces through the sixth, and, in some cases, for higher orders, has been exhaustively carried out; yet, except for certain surfaces generated by circles, such as surfaces of revolution and annular surfaces, and for surfaces containing a doubly infinite system of conics; i. e., the Steiner surfaces, the properties of surfaces generated by systems of conics has received little consideration. To determine some of the properties of certain of those surfaces and of the systems of conics lying on them is the object of this paper.

Among the leading articles dealing with this subject which have appeared, I may mention Koenigs'* paper on surfaces multiply generated by conics, also Stuyvaert's† paper on the properties of systems of conics determined by the condition of intersecting given fixed curves. Bertini‡ and Nugteren§ have considered special cases of the problem considered by Stuyvaert. Emil Weyr‖ has considered the problem of constructing the tangent planes to a surface along an arbitrary conic of a system lying on it.

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* "Determination de toutes les surfaces plusieurs fois engendrées par des coniques." Annales de L'Ecole Normale Superieure, Series 3, No. V.

† "Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre." Dissertation, Gand, 1902.

‡ "Sulle curve gobbe razionali del quinto ordine" in the Collectanea Mathematica in memoriam D. Chelini Mediolani, 1881, pp. 313–326.

§ "Rationale Ruimtekrommen van de fijde Orde." Dissertation, Utrecht, 1902.

‖ "Zur Theorie der Flächen, welche eine Schaar von Kegelschnitten enthalten." Monatshefte für Mathematik und Physik, Vol. II.

I. *On the system of tangent planes along a conic.*

The parametric equations of any surface containing a system of conics can be put in the form:

$$\chi_i = \theta_i(u) + 2v\,\phi_i(u) + v^2\psi_i(u) \qquad i = 1, 2, 3, 4.$$

The tangent plane to the surface at any point (u, v) is:

$$\begin{vmatrix} \chi_1 & \chi_2 & \chi_3 & \chi_4 \\ \theta_1 + v\phi_1 & \theta_2 + v\phi_2 & \theta_3 + v\phi_3 & \theta_4 + v\phi_4 \\ \phi_1 + v\psi_1 & \phi_2 + v\psi_2 & \phi_3 + v\psi_3 & \phi_4 + v\psi_4 \\ \theta_1' + 2v\phi_1' + v^2\psi_1' & \theta_2' + 2v\phi_2' + v^2\psi_2' & \theta_3' + 2v\phi_3' + v^2\psi_3' & \theta_4' + 2v\phi_4' + v^2\psi_4' \end{vmatrix} = 0,$$

in which θ_i', ϕ_i' and ψ_i' denote derivatives with respect to u.

This equation is of fourth degree in v. Hence, in general, the tangents to the surface along a fixed conic $u =$ const. form a developable of class four. This developable has the plane of the given conic for double plane. For it is easily seen that, at each of the points of intersection of the given conic with the plane of the consecutive conic $u + du$, the plane of the given conic is tangent to the surface.

If, however, for all values of u, this developable reduces to one of class three, then for some value of v the minors of $\chi_1\ \chi_2\ \chi_3$ and χ_4 in the above determinant must all vanish. This value of v is, in general, a function of u, say $v = f(u)$, but on replacing v by $v + f(u)$, we may, without altering the *form* of the equations of the surface, reduce this value to $v = 0$. Suppose this done. It then follows that

$$\begin{Vmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \theta_1' & \theta_2' & \theta_3' & \theta_4' \end{Vmatrix} \equiv 0$$

for all values of u.

It is thus seen that the surface belongs to one or the other of two classes:

a_1. The conics all pass through a fixed point.
a_2. The conics all touch a fixed curve.

The equations of a surface of the kind a_2 may be written in the form:

$$\chi_i = \theta_i + 2v\,\theta_i' + v^2\psi_i. \qquad i = 1, 2, 3, 4.$$

The analogy of the surfaces a_1 to cones and of the surfaces a_2 to developable surfaces is at once evident.

To the curve:

$$\chi_i = \theta_i(u) \qquad i = 1, 2, 3, 4$$

which is touched by all the conics of the system on a surface a_2. Darboux has given the name of "edge of regression." The analogy to the edge of regression of a developable is obvious. It is, in general, a double curve on the surface. Indeed, the only surfaces on which it is not nodal are easily seen to be those through each point of which pass two conics of the system. Koenigs* has shown that the only surfaces through every point of which pass two conics of the system are those which contain a doubly infinite system of conics; i. e., the Steiner surface and its degenerate cases, the ruled cubic, quadric and plane.

In the determinant equation of the tangent plane, the minors of χ_1, χ_2, χ_3 and χ_4 may all contain a common factor quadratic in v. The tangent planes along an arbitrary conic then envelope a quadric cone. This happens when the surface belongs to one of the following classes:

b_1. The conics all pass through two fixed points.

b_2. The conics all pass through a fixed point and touch a fixed curve.

b_3. The conics all touch a fixed curve (which may be either proper or composite) at two points.†

b_4. The conics all have contact of the second order with a fixed curve. In the case of the surfaces b_4 the quadratic factor common to the four minors is the square of a linear factor. The equations of such a surface may be written in the form:

$$\chi_i = \theta_i + v\,\theta_i' + v^2\left(\frac{\theta_i''}{2} + a\,\theta_i' + b\,\theta_i\right) \qquad i = 1, 2, 3, 4$$

a and b being constants.

All the conics have three point contact with the curve:

$$\chi_i = \theta_i(u) \qquad i = 1, 2, 3, 4$$

This curve is, in general, triple on the surface; at each of its points the

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[*] Loc. cit.

[†] Enneper writing in the Zeitschrift für Mathematik und Physik, in 1869, and, following him, Cosserat in the Annales des Faculté des Sciences de Toulouse, in 1889, have inferred that the tangents along a conic may envelope a quadric cone when the given conic meets the consecutive conic at only one of its intersections with the plane of the latter. Since, however, the plane of the given conic would have to be tangent to the surface at the second intersection, the developable of tangents would have to be of third class.

three tangent planes coincide. No surface of this kind is of order low as five. A simple example of such a surface is:

$$\chi_1(\chi_3^2-\chi_2\chi_4)^3+3\chi_2(\chi_3^2-\chi_2\chi_4)^2(\chi_2^2-\chi_1\chi_3)+3\chi_3(\chi_3^2-\chi_2\chi_4)(\chi_2^2-\chi_1\chi_3)^2$$
$$+\chi_4(\chi_2^2-\chi_1\chi_3)^3=0$$

on which the twisted cubic:

$$\chi_3^2-\chi_2\chi_4=0 \qquad \chi_2^2-\chi_1\chi_3=0$$

has three point contact at each of its points with a conic in the plane:

$$\chi_1 u^3+3\chi_2 u^2+3\chi_3 u+\chi_4=0.$$

Since a conic is not completely determined by the conditions of having contact of first or second order with a given curve at a given point, the curve

$$\chi_i=\theta_i(u) \qquad i=1,\ 2,\ 3,\ 4$$

may have contact of the first or second order at each of its points with several conics of the system. Thus, the line $\chi_1=\chi_2=0$ on the surface,

$$\chi_1^4\chi_3^2+\chi_1^4\chi_2\chi_3+2\chi_1^2\chi_2^2\chi_3\chi_4+\chi_1\chi_2^4\chi_4+\chi_2^4\chi_4^2=0,$$

is touched at each of its points by two conics of the system lying in the pencil of planes through the line.

Similarly, on the surface:

$$\chi_1(\chi_3^2-\chi_2\chi_4)^6+3\chi_2(\chi_3^2-\chi_2\chi_4)^4(\chi_2^2-\chi_1\chi_3)^2+3\chi_3(\chi_3^2-\chi_2\chi_4)^2(\chi_2^2-\chi_1\chi_3)^4$$
$$+\chi_4(\chi_2^2-\chi_1\chi_3)^6=0,$$

the twisted cubic:

$$\chi_3^2-\chi_2\chi_4=0 \qquad \chi_2^2-\chi_1\chi_3=0$$

has contact of the second order at each of its points with two conics belonging to the same system.

It will presently be shown that, in general, on a surface generated by conics there are points through which pass two consecutive conics. These points may, by analogy with the corresponding singularity on ruled surfaces, be called pinch-points. As in the case of ruled surfaces these points are uniplanar points. Along a conic passing through a pinch-point, the developable of tangents reduces to class three. The condition for such a conic is, therefore, that, for a particular value of u, the determinant given on page 100 reduce to one of third degree in v.

There exist, also, surfaces on which discrete conics meet the conics consecutive to them in two points or which meet two consecutive conics in a uniplanar triple point. The tangents to the surface along such a conic envelope a quadric cone.

Finally, two (or more) consecutive conics may be coplanar. The plane of such a conic touches the surface all along the conic.

II. *On the determination of the properties of surfaces containing unicursal systems of conics.*

In the equations

$$\chi_i = \theta_i(u) + 2v\,\phi_i(u) + v^2\,\psi_i(u) \qquad i = 1, 2, 3, 4$$

the expressions $\theta_i(u)$, $\phi_i(u)$ and $\psi_i(u)$ may, when the system is unicursal, be taken to be polynomials in u. The conics therefore lie in the planes of a developable whose equations may be put in the form:

$$L_1 u^m + m\,L_2 u^{m-1} + \frac{m(m-1)}{2} L_3 u^{m-2} + \ldots\ldots\ldots + L_{m+1} = 0$$

in which — as throughout this paper — $L_i = 0$ is the equation of a plane. They also lie on the surfaces of a system of quadrics of the form:

$$Q_1 u^n + n\,Q_2 u^{n-1} + \frac{n(n-1)}{2} Q_3 u^{n-2} + \ldots\ldots\ldots + Q_{n+1} = 0,$$

$Q_i = 0$ being the equation of a quadric surface.

Any conic of the system is the intersection of the plane and quadric determined by the same value of u. The surface on which the sytem lies is found by eliminating u between the two equations. Its order, M, is, in general, $2m + n$. When, however, any plane is a component of its corresponding quadric the order of the surface is reduced. If, in this case, $n \geqq m$, then the system of quadrics can be replaced by one of lower degree in u. For, let $u = 0$ be the value of u for which the plane is a component of the quadric. This is no further restriction on the system. We then have:

$$\begin{aligned} L_1 u^m + m\,L_2 u^{m-1} + \ldots\ldots\ldots + L_{m+1} &= 0 \\ Q_1 u^n + n\,Q_2 u^{n-1} + \ldots\ldots\ldots + L_{m+1} L' &= 0. \end{aligned}$$

On multiplying the first equation by L' and subtracting, then dividing the resulting equation by u, the required system is obtained. We may, therefore, without loss of generality, suppone one or the other of the relations

$$M = 2m + n \quad \text{or} \quad n < m$$

satisfied.

At any point of the nodal curve of the surface defined by the system, there are two values of u for which both equations of the system are satisfied. The nodal curve is therefore determined by the conditions that those equations have two common solutions.

At a pinchpoint the two common solutions of the given equations are equal. At such a point, therefore, in addition to the equations defining the system, we have:

$$L_1 u^{m-1} + (m-1) L_2 u^{m-2} + \dots\dots + L_m = 0$$
$$Q_1 u^{n-1} + (n-1) Q_2 u^{n-2} + \dots\dots + Q_n = 0.$$

Since this is only four conditions on u and the three ratios of the coordinates of a point, there exist on the surface, in general, a finite number of pinchpoints. When a curve is enveloped by the conics of the system every point of it is a pinchpoint. The edge of regression, when it exists, is, therefore, determined by these four equations.

When, at any point, the above four equations are satisfied and also

$$L_1 u^{m-2} + (m-2) L_2 u^{m-3} + \dots\dots + L_{m-1} = 0$$
$$Q_1 u^{n-2} + (n-2) Q_2 u^{n-3} + \dots\dots + Q_{n-1} = 0$$

then three consecutive conics meet at that point. When these six equations are satisfied at every point of a curve, then that curve has contact of the second order at each of its points with the conics of the system.

The condition that two consecutive conics be coplanar, is that, for some values of u, the planes:

$$L_1 u^{m} + m L_2 u^{m-1} + \dots\dots + L_{m+1} = 0$$
$$L_1 u^{m-1} + (m-1) L_2 u^{m-2} + \dots\dots + L_m = 0$$

be identical.

Three consecutive conics will be coplanar if, in addition:

$$L_1 u^{m-2} + (m-2) L_2 u^{m-3} + \dots\dots + L_{m-1} = 0$$

is identical with each of the other two; and similarly for any number of consecutive coplanar conics.

There may exist on the surface certain nodal straight lines which are not determined by the condition that the equations of the system have two common solutions. Let, for example:

$$Q_{n+1} \equiv L'^2 + L_{m+1} L''$$
$$Q_n \equiv L_{m+1} L''' + L' L^{IV} + L_m L''.$$

Then $L' = L_{m+1} = 0$ is a double line on the surface, although at an arbitrary point upon it the equations of the system have only one common solution.

a. *Systems of conics on cubic surfaces.*

The residual intersection of the plane of any conic with the surface is a straight line. Similarly, an arbitrary plane through an arbitrary straight line on the surface meets the surface in a conic. When the surface is not ruled, therefore, there exist on it twenty-seven systems of conics in the planes through the twenty-seven lines. The equations of any one of these systems may be written in the form:

$$L_1 u + L_2 = 0$$
$$Q_1 u + Q_2 = 0.$$

The equation of the surface is then:

$$L_1 Q_2 - L_2 Q_1 = 0.$$

When the line $L_1 = L_2 = 0$ meets the twisted quartic curve $Q_1 = Q_2 = 0$, all the conics pass through a fixed point. This point is then a node on the surface. Conversely, the conics in the pencil of planes on any line through a node all pass through the node. It follows that the class of the developable of tangent planes along any conic is reduced by unity for every node through which the conic passes.

When the cubic is ruled, an arbitrary tangent plane meets it in a rectilinear generator and a conic. There exist, therefore, on the surface, a double infinity (∞^2) of conics; namely, those in the double infinity of tangent planes. The planes of any developable of tangents to the surface cut from the surface a system of conics. This system of conics is of the same genus as the developable, for the conics of the system are in one to one correspondence with the planes of the developable. On the ruled cubic, therefore, there exist systems of conics of every genus, whereas, on the unruled cubic, the only possible systems are unicursal.

The tangent planes to the ruled cubic along any conic lying on it form a developable of class three. All the conics lying in the planes of any developable of tangents and which do not all pass through a fixed point must, therefore, touch a fixed curve. It will be shown in the case of the Steiner surface, of which the ruled cubic is a particular case, that any curve whatever on the surface is touched by a system of conics provided only that the parametrically corresponding curve is the envelope of a system of lines.

There are only two systems of conics on the ruled cubic along which the tangents to the surface envelope a quadric cone. These are the systems through the torsal generators. The conics of each system have two consecutive common points at the pinchpoints.

b. *Unicursal systems lying on quartic surfaces.*

1. *Developable of planes form a linear pencil.*

The equations of the system are:

$$L_1 u + L_2 = 0,$$
$$Q_1 u^2 + 2 Q_2 u + Q_3 = 0.$$

The equation of the surface is, therefore:

$$Q_1 L_2^2 - 2 Q_2 L_1 L_2 + Q_3 L_1^2 = 0.$$

The nodal curve is the line $L_1 = L_2 = 0$. The surface may also have one or two additional nodal lines obtained by the method shown on page 104. These lines necessarily meet $L_1 = L_2 = 0$. When the surface has one additional nodal line it is a special case of the quartic surfaces having a nodal conic and therefore also has on it systems of conics whose planes envelope quadric cones. When the surface has two additional nodal lines, it is either ruled or a Steiner surface according as these two lines do not or do intersect.

The four intersections of the line $L_1 = L_2 = 0$ with the surface $Q_2^2 - Q_1 Q_3 = 0$ are the pinchpoints of the system. In the cases where the surface contains other systems of conics, these other systems may determine pinchpoints through which do not pass two consecutive conics of this system.

All the conics of the system may touch the line $L_1 = L_2 = 0$. This happens when, for all values of u, the quartic curve

$$Q_1 u + Q_2 = 0$$
$$Q_2 u + Q_3 = 0$$

meets that line. The line $L_1 = L_2 = 0$ is then the edge of regression of the surface for this system of conics.

2. *The planes of the conics envelope a quadric cone.*

The equations are of the form:

$$L_1 u^2 + 2 L_2 u + L_3 = 0$$
$$L_1 L_4 u + Q_2 = 0.$$

The nodal curve is the intersection of the surfaces:

$$L_4 = 0 \qquad Q_2 = 0.$$

It is either a proper conic or two intersecting straight lines, either distinct or consecutive. Conversely, any quartic surface whose complete nodal curve is of any of these three kinds has on it a system of conics of the above form. When however, the nodal lines are skew, either distinct or consecutive, the surface is ruled and has on it no system of conics.

The surface may have one additional nodal line. It is then ruled or a Steiner surface according as the nodal conic is not, or is, composite.

Whenever the nodal conic is not composite it may be projected into the absolute. The surface is then a *cyclide*. All of these surfaces, therefore, whose nodal conics are not composite are projections of the cyclides.

The pinchpoints of the system are the four intersections of the nodal conic $L_4 = Q_2 = 0$ with the cone $L_2^2 - L_1 L_3 = 0$. The surfaces for which the nodal conic is the edge of regression of the system may be determined by putting:

$$Q_2 \equiv L_2^2 - L_1 L_3 + L_4 L_5 .$$

When $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ all contain the same line, the developable of the planes of the conics reduces to a linear pencil counted twice. Two conics of the system are here coplanar with the conics consecutive to them. The equations of such a system may be reduced to the form:

$$\begin{aligned} L_1 u^2 + L_2 &= 0 \\ L_1 L_3 u + Q_2 &= 0. \end{aligned}$$

All the conics of the system pass through the two points $L_1 = L_2 = Q_2 = 0$. Conversely, if two conics of any system on a quartic which is not a Steiner surface lie in the same plane, then all the conics pass through two fixed points and, if the system is unicursal, its equations can be put in the above form. When the surface is a Steiner surface, however, this theorem is not true, since two conics of the same system may pass through every point of such a surface.

When the planes of the conics form a cubic developable the surface is either a ruled quartic or a Steiner surface, since the nodal curve is a cubic.

3. *Systems of conics on ruled quartics.*

The ruled quartic is the ruled surface of highest degree having on it a system of conics. Both the system and the surface must be rational.

The planes of the system must either form a linear pencil, or touch a quadric cone or form a developable of class three. In the first case the surface has two nodal rectilinear directrices and the axis of the pencil is a double generator. In the second case the nodal curve is a rectilinear directrix and a proper conic. In the third case the nodal curve is either a triple rectilinear directrix or a double cubic.

The conics can touch a curve on the surface in two cases only; first, when their planes form a linear pencil about a cuspidal generator and, second, when the nodal curve is a proper cubic and the surface is itself developable.

4. *Systems of conics on the Steiner surface.*

The intersection of any tangent plane with the surface is a quadrinodal quartic and therefore breaks up into two conics. The developable of tangents to the surface along any such conic can not be of more than third class; for if it were of fourth class the plane of the conic would have to be a double tangent plane to the surface. The conics of any system, therefore, which is chosen so that the conics do not all pass through a fixed point, all touch a fixed curve.

It is well known that the equations of the surface can be put into the form:

$$\chi_i = a_i + b_i u + c_i v + d_i u^2 + e_i u v + f_i v^2 \qquad i = 1, 2, 3, 4$$

To the points of any line:

$$\alpha u + \beta v + \gamma = 0$$

in the (u, v) plane correspond the points of a conic on the surface. To any curve on the surface corresponds another curve

$$F(u, v) = 0.$$

The system of tangents to $F = 0$ determine on the surface a system of conics touching the corresponding curve. Hence, any curve on the surface such that the corresponding curve in the (u, v) plane is the envelope of a system of lines is itself the envelope of a system of conics.

No curve on the surface is either osculated or touched twice by a system of conics since the corresponding curve $F = 0$ can not be osculated or touched twice by a system of lines.

c. *Quintic surfaces.*

1. *The planes of the conics form a linear pencil.*

The equations of the system are of the form:

$$L_1 u + L_2 = 0,$$
$$Q_1 u^3 + Q_2 u^2 + Q_3 u + Q_4 = 0.$$

The nodal curve is the triple line $L_1 = L_2 = 0$. The surface may have one or two additional nodal lines under the conditions mentioned on page 104. It can not have three double lines because a quintic surface with a nodal curve of order six must be ruled* and a ruled quintic can not have on it a family of proper conics. When the surface has two double lines it contains another system of conics, whose planes form a developable of class three. The double lines may, in particular, be consecutive with each other or with the triple line.

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* See Picard in Crelle's Journal, Vol. 100.

The eight pinchpoints of the system are the intersections of the line $L_1 = L_2 = 0$ with the envelope of the system of quadrics. All the conics of the system will touch the triple line when, for all values of u, the quartic curves:

$$Q_1 u^2 + 2\, Q_2 u + Q_3 = 0$$
$$Q_2 u^2 + 2\, Q_3 u + Q_4 = 0$$

meet that line. Two tangent planes will then coincide at each point of the triple line, the third being torsal along the line.

2. *The planes of the conics envelope a quadric cone.*

The equations of the system are:

$$L_1 u^2 + 2\, L_2 u + L_3 = 0,$$
$$Q_1 u + Q_2 = 0.$$

The nodal curve is the (proper or composite) quartic curve $Q_1 = Q_2 = 0$. The surface may also contain an additional nodal line under the conditions mentioned on page 104.

The eight pinchpoints of the system are the intersections of the nodal quartic with the cone

$$L_2^2 - L_1 L_3 = 0.$$

Any curve which is enveloped by all the conics of the system lies on each of the quadrics:

$$Q_1 = 0, \qquad Q_2 = 0, \qquad L_2^2 - L_1 L_3 = 0.$$

The component so enveloped may be either a conic or a cubic or a quartic curve.

The nodal curve may also be touched twice by all the conics of the system. When this happens we may, without restricting the surface, put $Q_1 \equiv L_2^2 - L_1 L_3$, Q_2 being arbitrary. The curve thus enveloped is, in general, a quartic; but it may break up into two conics each of which is touched once.

As in the corresponding case of quartic surfaces, the system of planes of the conics may degenerate into a linear pencil, each plane of which contains two conics of the system. Two conics of such a system are coplanar with the conics consecutive to them and the equations of the system may be put into the form

$$L_1 u^2 + L_2 = 0,$$
$$Q_1 u + Q_2 = 0.$$

3. *Developable of the third class.*

It will be convenient to distinguish two cases. First, let

$$L_1 u^3 + 3 L_2 u^2 + 3 L_3 u + L_4 = 0,$$
$$L_1 L_5 u + L_4 L_6 = 0.$$

(The case in which the family of quadrics is of the form:

$$(3 L_3 L_5 - L_4 L_6) u + L_4 L_5 = 0$$

is a sub-case of the above.)

The line $L_5 = L_6 = 0$ is a triple line and $L_1 = L_6 = 0$, $L_4 = L_5 = 0$ are double lines on the surface. The surface therefore also contains another system of conics; namely, those in the pencil of planes through the triple line.

The system has eight pinchpoints: the four intersections of $L_5 = L_6 = 0$ with the envelope of the system of planes; the two points $L_1 = L_6 = 3 L_3^2 - 4 L_2 L_4 = 0$ and the two points $L_4 = L_5 = 3 L_2^2 - 4 L_1 L_3 = 0$. It can not envelope a component of the nodal curve for the surface enveloped by the planes:

$$L_1 u^3 + 3 L_2 u^2 + 3 L_3 u + L_4 = 0$$

can not have a rectilinear directrix.

The other systems are those of the form:

$$L_1 u^3 + (L_2 - L_1) u^2 + (L_3 - L_4) u + L_4 = 0,$$
$$L_1 L_6 u^2 + [(L_2 + L_3) L_7 - L_1 L_6 - L_4 L_5] u + L_4 L_5 = 0.$$

By combining these two equations we obtain:

$$L_1 L_5 u^2 + (L_2 L_5 - L_1 L_5 - L_1 L_6) u + L_3 L_5 + L_1 L_6 - (L_2 + L_3) L_7 = 0.$$

At any point of the nodal curve the two quadratic equations have two common roots. The nodal curve is thus seen to be the intersection, other than $L_5 = L_6 = 0$ of

$$L_5 L_6 = L_5 L_7 + L_6 L_7,$$
$$L_3 L_5 L_6 + L_1 L_6^2 = L_4 L_5^2 + (L_2 + L_3) L_6 L_7.$$

It is a quintic curve with a triple point at $L_5 = L_6 = L_7 = 0$. This nodal quintic may decompose into a quartic with a node at $L_5 = L_6 = L_7 = 0$ and a straight through the node meeting the quartic again, or into a cubic and two straight lines both meeting the cubic at the same point and each meeting it again or into a conic and three concurrent lines each meeting the conic.

In each case where the curve is composite, a pencil of quadrics can be passed through the entire nodal curve exclusive of one nodal line. The residual intersections of such a system of quadrics with the surface is a system of conics whose planes envelope a quadric cone.

The system has eight pinchpoints. One conic of the system may be coplanar with the conic consecutive to it. The developable of the planes of the conics must then degenerate into a cone of third class.

The conics can not envelope the nodal curve. To see this first consider the case of a proper quintic. The conics can not be doubly tangent to this quintic, for if they were, each generator of the envelope of the planes of the conics would have to be a bisecant of the quintic. This is impossible, for of the two intersections of each generator of that developable with the cone,

$$L_5 L_6 = L_5 L_7 + L_6 L_7,$$

on which the quintic lies, at least one must lie on the residual curve of intersection of the surfaces. It follows that the conics do not envelope the quintic at all. For, the two conics through an arbitrary point of the quintic would be consecutive, yet each conic would have to meet on the quintic two conics not consecutive with it. When the quintic breaks up into a quartic and a line, a similar proof holds for the quartic. The line can not be enveloped since it is met by each conic but once.

When the quintic breaks up into a cubic and two straight lines, we may take these lines, since they intersect, for fundamental lines and a point on the cubic for fundamental point in a quadratic-quadratic Cremona transformation which transforms the surface into a quartic surface and the system of conics into a system of conics. If the original system touched the cubic, the transformed system would touch the conic into which the cubic is transformed. When, however, the inverse transformation is performed on such a system of conics on a quartic, the planes of the conics are seen to envelope a surface of class two instead of class three as here supposed.

The case of a nodal conic may be disposed of like that of a nodal cubic. The nodal conic may, however, be a cusp locus on the surface. When this happens, all the conics of the system meet this conic in a fixed point and the conic is nodal because two conics of the system coincide with it throughout.

III. *Systems of genus greater than zero.*

Any algebraic system of conics is determined by three equations of the form:

$$L_1 + L_2 u + L_3 v + \ldots\ldots\ldots = 0$$
$$Q_1 + Q_2 u + Q_3 v + \ldots\ldots\ldots = 0$$
$$f_n(u, v) = 0$$

wherein u and v are parameters and $f_n(u, v)$ is a polynomial in u and v of degree n and with constant coefficients. To each point on $f_n = 0$, considered as a curve in the (u, v) plane corresponds a conic of the system and conversely. The genus of the system is, therefore, equal to that of $f_n = 0$. Since the genus of the system is supposed greater than zero, we must have $n \geqq 3$.

If the equation determining the planes of the conics is considered as the equation of a curve in the (u, v) plane which meets $f_n(u, v) = 0$ in m points and if the equation of the quadrics, similarly considered, determines a curve which meets $f_n = 0$ in m' points, then the order of the surface determined by the system is, in general, $2m + m'$.

For certain pairs of values of u and v the corresponding quadrics will be composite. When such a point (u, v) lies on $f_n = 0$, and when the corresponding plane coincides with a component of the quadric, then the equation of the plane is a factor of the equation of the surface. As in the unicursal systems, when a plane is a component of its corresponding quadric, the equation of the system of quadrics may frequently be reduced.

It is usually true that only one conic of the system lies in an arbitrary plane of its developable. We may then take two of the non-homogeneous point coordinates of the planes for u and v. Since the coordinates (α, β, γ) of the planes of the developable satisfy the equations:

$$\gamma = \frac{F_m(\alpha, \beta)}{F_{m-1}(\alpha, \beta)} \qquad f_n(\alpha, \beta) = 0$$

it is seen that the equation of the planes of the conics may, in this case, be written:

$$L_1 + L_2 u + L_3 v + \frac{F_m}{F_{m-1}}, \; L_4 = 0.$$

At an arbitrary point of the nodal curve of the surface determined by the system, there are two pairs of values of (u, v) which satisfy the three given equations. The nodal curve is, therefore, determined by the condition that the three equations determine two common points in the (u, v) plane.

The pinchpoints are determined by the condition that the three given curves touch at a common point. This is five conditions on the five quantities u, v and the ratios of the coordinates of a point. The system has, therefore, in general, a finite number of pinchpoints. When, at a point, a conic meets two consecutive ones, the corresponding three (u, v) curves have contact of the second order at a common point. A curve which is touched by all the conics of the system is therefore determined by the condition that the three (u, v) curves touch at a common point; and one which has contact of the second order with all the conics, by the condition that the three (u, v) curves have contact of the second order.

Additional nodal right lines, through an arbitrary point of which passes only one conic of the system, may arise as in the case of unicursal systems and under similar conditions.

When a point (u, v) is a double point of $f_n(u, v) = 0$, the corresponding conic is, in general, a double conic on the surface. Moreover, if the multiple point is a cusp, the conic is a cusp locus, if the multiple point is a tacnode two sheets of the surface touch along the conic and similarly for higher singularities. This is seen by putting:

$$u - u_0 = t^n \qquad v - v_0 = a_1 t + a_2 t^2 + \ldots\ldots\ldots$$

and determining the form of the surface in the neighborhood of the conic $t = 0$.

When, as is the case in the surfaces in which we shall be interested, the first two of the three given equations are linear in u and v, it is obvious that, at any point of the nodal curve other than that determined by multiple points of $f_n = 0$ and the right lines mentioned above, these two lines in the (u, v) plane must coincide. The nodal curve is, therefore, determined by the equations:

$$\frac{L_1}{Q_1} = \frac{L_2}{Q_2} = \frac{L_3}{Q_3}.$$

It is, in general, of order seven and of multiplicity n on the surface.

a. *Quartic surfaces.*

The system of conics:

$$L_1 + u L_2 = 0$$
$$L_1^2 + v Q_2 = 0$$
$$(a_0 v^2 + b_0 v + c_0) u^4 + (a_1 v^2 + b_1 v) u^3 + (a_2 v^2 + b_2 v) u^2 + a_3 v^2 u + a_4 v^2 = 0$$

determines a quartic surface. The system is of genus one, and the surface has no nodal curve, unless $f_n = 0$ has an additional double point, in which case the genus of the system is reduced to zero. In each plane of the pencil:

$$L_1 + u L_2 = 0$$

lie two conics of the system which touch at each of the points:

$$L_1 = L_2 = Q_2 = 0.$$

These points are tacnodal points on the surface. For four values of u the coplanar conics are consecutive. These four values of u are determined by the four tangents to $f_n = 0$ which are parallel to the v axis.

On the Steiner surface are systems of conics of any genus whatever, lying in the planes of the developables of tangents to the surface.

b. *Quintic surfaces.*

The system of conics satisfying the equations:

$$\begin{gathered} L_1 + u L_2 = 0 \\ L_1^2 + v Q_2 = 0 \\ (a_0 v^2 + b_0 v + c_0) u^5 + (a_1 v^2 + b_1 v + c_1) u^4 + (a_2 v^2 + b_2 v) u^3 + (a_3 v^2 + b_3 v) u^2 \\ + a_4 v^2 u + a_5 v^2 = 0 \end{gathered}$$

lies on a quintic surface. Two conics of the system lie in each plane of the pencil:

$$L_1 + u L_2 = 0$$

and touch at each of the tacnodal points:

$$L_1 = L_2 = Q_2 = 0.$$

The system is, in general, of genus two, but the genus may reduce by the appearance of additional double points in $f_n(u, v) = 0$. The surface has no nodal curve when the system of conics is of genus two, but, with decreasing genus, it has one or two nodal conics. These nodal conics may be cuspidal, consecutive, etc.

The number of consecutive coplanar conics is, at most, six.

The system of conics

$$\begin{gathered} L_1 + u L_2 = 0 \\ L_1 L_3 + v Q_2 = 0 \\ (a_0 v^2 + b_0 v + c_0) u^3 + (a_1 v^2 + b_1 v + c_1) u^2 + (a_2 v^2 + b_2 v) u + a_3 v^2 = 0 \end{gathered}$$

is of genus one and lies on a quintic surface.

The conic $L_3 = Q_2 = 0$ is nodal, or, in particular, cuspidal, on the surface. The surface may have an additional nodal line. $L_2 = L_4 = 0$ is such a line when:

$$a_0 = 0 \qquad Q_2 \equiv L_4^2 + L_2 L_5 \qquad b_0 L_5 + a_1 L_3 \equiv \alpha L_2 + \beta L_4$$

where α and β are constants.

The two conics in an arbitrary plane

$$L_1 + u L_2 = 0$$

meet twice on the nodal conic and also at each of the points

$$L_1 = L_2 = Q_2 = 0.$$

Four conics of the system are coplanar with the conics consecutive to them.

When the plane determined by an arbitrary conic of the system does not contain another conic of the system, the planes of the conics envelope a cone of class three. Since the imposition on this cone of the condition of being unicursal is equivalent to bringing an additional nodal conic on the surface, it is seen that the nodal curve is of order three.

This nodal curve can not be a proper cubic, however, for the surface would then be rational as is seen by letting correspond to any point of it the point in which a fixed plane is pierced by the bisecant to the nodal cubic through the given point and conversely. Neither can the nodal curve be a proper conic and a line for such a surface is easily seen to be either rational or composite.

There do exist, however, quintics determined by systems of conics, whose nodal curve consists of three concurrent straight lines. Each conic of the system intersects each nodal line and, since it must meet four other conics of the system, passes through the vertex of the cone determined by the planes of the conics.

Taking two of the lines for fundamental lines, and the vertex of the cone for fundamental point in a quadratic-quadratic Cremona transformation, the surface is transformed into a ruled quartic of genus one, having the fundamental point for simple point. The tangent cone to the quartic at this point is of the form:

$$L_1 u + L_2 v + L_3 = 0$$
$$f_3(u, v) = 0.$$

It is easily seen that an infinite number of cones of class three exist which are tangent to the quartic and whose equations are of the form:

$$L_1' u + L_2' v + L_3' = 0,$$

wherein u and v are joined by the same cubic relation $f_3 = 0$.

Performing, now, the inverse transformation, we have the system of conics determined by:

$$L_1 u + L_2 v + L_3 = 0$$
$$Q_1 v + Q_2 v + Q_3 = 0$$
$$f_3(u, v) = 0.$$

These equations can be still further specialized. When the surface is a quintic, there must be four pairs of values of u and v satisfying $f_3 = 0$ for which the first equation is a factor of the second. It is easily seen that three of these four points in the (u, v) plane are collinear. Hence, except in the particular cases in which some of them are consecutive, the equations of the system may be written:

$$L_1 u + L_2 v + L_3 = 0$$
$$L_1(L_4 + L_3) u + L_3(L_4 - L_1) = 0$$
$$a u^2 v + b u v^2 + c(u^2 - u) + d u v + e v^2 + f v = 0.$$

The equation of the surface is:

$$a L_2 L_3^2 (L_1 - L_4)^2 - b L_1 L_3^2 (L_1 - L_4)(L_1 + L_3) + c L_2^2 L_4 (L_3 + L_4)(L_1 - L_4)$$
$$+ d L_1 L_2 L_3 (L_1 - L_4)(L_3 + L_4) - e L_1^2 L_3 (L_1 + L_3)(L_3 + L_4) + f L_1^2 L_2 (L_3 + L_4)^2 = 0.$$

The three nodal lines are:

$$L_1 = L_4 = 0$$
$$L_3 = L_4 = 0$$
$$L_1 + L_3 = L_3 + L_4 = 0.$$

These lines may become consecutive.

All the conics of the system touch $L_2 = 0$ at $L_1 = L_2 = L_3 = 0$. On each nodal line are four pinchpoints of the system. The conics obviously can not envelope any of the nodal lines since they meet each line in only one point.

U. S. NAVAL ACADEMY, *June* 10, 1906.

On the Canonical Forms and Automorphs of Ternary Cubic Forms.

BY L. E. DICKSON.

Gordan has given* a complete set of canonical types of ternary cubic forms and has determined the algebraic irrationalities occurring in the reducing linear transformations. There does not seem to be at hand a reduction theory in which the coefficients of the form and those of the reducing transformations belong to a given field F. The case in which F has the modulus 3 is essentially different from the contrary case and will be treated in the present paper. After treating the reduction problem rationally in the initial field, we consider, in §§ 19–20, reductions involving irrationalities and obtain eleven ultimate canonical forms. This result for modular fields is in contrast to Gordan's results for the field of all complex numbers.

1. Let F be a field having modulus 3, and let

$$f \equiv \sum_{i}^{1,2,3} a_i x_i^3 + \sum_{i,j}^{1,2,3} c_{ij} x_i^2 x_j + b x_1 x_2 x_3 \tag{1}$$

have its coefficients in F. The Hessian of f is

$$\Sigma A_i x_i^3 - Q \{\Sigma c_{ij} x_i^2 x_j + b x_1 x_2 x_3\}, \tag{2}$$

where

$$Q \equiv b^2 - c_{21} c_{31} - c_{12} c_{32} - c_{13} c_{23}, \tag{3}$$

$$A_i \equiv c_{ij}^2 c_{ki} + c_{ik}^2 c_{ji} - b c_{ij} c_{ik} \qquad (i, j, k = 1, 2, 3). \tag{4}$$

We infer that Q is an invariant of f and that

$$\sum_{i}^{1,2,3} (A_i + Q a_i) x_i^3 \tag{5}$$

is a covariant of f. These facts indicate the exceptional character of the case in which the field has modulus 3.

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* *Transactions American Mathematical Society*, vol. 1 (1900), p. 403.

2. Suppose first that the c_{ij} are not all zero. By an evident transformation, we may set $c_{12}=1$. Applying in turn* B_{12t}, $B_{23t'}$, $B_{13t''}$, we may make $c_{21}=c_{13}=b=0$. The resulting form may be given the notation

$$(6) \qquad f(x) \equiv x_1^2 x_2 - Q x_3^2 x_2 + R x_2^2 x_3 + S x_3^2 x_1 + \Sigma a_i x_i^3.$$

Under the transformation

$$(7) \qquad x_1 = y_1 - tRy_3, \quad x_2 = ty_1 + y_2 + t^2 R y_3, \quad x_3 = y_3,$$

$f(x)$ becomes $f'(y)$, in which

$$(8) \qquad \begin{cases} R' = R, \ S' = S - tQ + t^3 R^2, \ a_1' = a_1 + t^3 a_2 + t, \\ a_2' = a_2, \ a_3' = a_3 + t^6 R^3 a_2 - t^3 R^3 a_1 - tRS - t^2 QR - t^4 R^3. \end{cases}$$

Henceforth, let F be the $GF[3^n]$. Then $S'=0$ requires

$$(9) \quad tQ - t^3R^2 = S, \ t^3Q^3 - t^9R^6 = S^3, \ \ldots, \ t^{3^{n-1}} Q^{3^{n-1}} - tR^{2 \cdot 3^{n-1}} = S^{3^{n-1}}.$$

The determinant of the coefficients of $t, t^3, \ldots, t^{3^{n-1}}$ equals

$$(10) \qquad \Delta \equiv Q^{\frac{1}{2}(3^n-1)} - R^{3^n-1}.$$

If $\Delta \neq 0$, $t, t^3, t^9, \ldots$ are uniquely determined by (9), and the resulting value of t^3 is seen to equal the cube of that of t, etc. Hence, if $\Delta \neq 0$, we may set $S=0$ in (6). Then, if $Q=0$, we multiply x_3 by a suitable mark and get

$$(11) \qquad x_1^2 x_2 + x_2^2 x_3 + \Sigma a_i x_i^3.$$

But if $Q \neq 0$, we apply B_{32} and make $R=0$. Then if Q is the square of a mark τ, we introduce $x_1 \pm \tau x_3$ and x_2 as new variables, and are led to the case treated in § 3. If Q is a not-square, we multiply x_1 by λ, and x_2 by λ^{-2}, and choose λ to specialize the new Q; there results

$$(12) \qquad x_1^2 x_2 - \nu x_3^2 x_2 + \Sigma a_i x_i^3 \quad (\nu \text{ a particular not-square}).$$

Next, let $\Delta = 0$. If $R=S=0$, (6) becomes

$$(13) \qquad x_1^2 x_2 + \Sigma a_i x_i^3.$$

If $R=0$, $S \neq 0$, we set $y_1 = x_3$, $y_2 = Sx_1$, $y_3 = S^{-2}x_2$, and obtain (11). Finally, let $R \neq 0$, so that, by (10), Q is a square $\neq 0$. Multiplying x_1 by λ, x_2 by λ^{-2}, and x_3 by $\lambda\mu$, and taking† $\mu^2 = Q^{-1}$, $\lambda^3 = R\mu$, we get

$$(14) \qquad x_1^2 x_2 - x_3^2 x_2 + x_2^2 x_3 + s x_3^2 x_1 + \Sigma a_i x_i^3.$$

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[*] In the usual notation, B_{12t} alters only x_1, replacing it by $x_1 + tx_2$.

[†] Any mark ρ of the $GF[3^n]$ is the cube of the mark $\rho^{3^{n-1}}$.

In view of (8), this can be transformed into a similar form with

(e) $$s' = s - t + t^3.$$

Now $t^3 - t = c$ is solvable in the $GF[3^n]$ if and only if

$$\phi(c) \equiv c + c^3 + c^9 + \dots + c^{3^{n-1}}$$

vanishes. Hence, if $\phi(s) = 0$, (14) is equivalent to a similar form with $s = 0$. To the latter apply $B_{32,-1}$; there results $(y_1^2 - y_3^2)y_2 + \Sigma$, which obviously falls under the case treated in § 3. Since $\phi^3 = \phi$, there remains the case $\phi(s) = \pm 1$. Under the transformation $x_1' = -x_1$, the sign of s in (14) is changed. Hence we may set $\phi(s) = +1$. But if $\phi(s') = \phi(s)$, then $\phi(s' - s) = 0$, and (e) is solvable for t in the $GF[3^n]$. *Hence we may restrict s in* (14) *to be a particular solution of $\phi(s) = 1$.* In case n is prime to 3, we may set $s = 1$.

3. Next, let every c_{ij} be zero. According as $b \neq 0$ or $b = 0$, we get

(15) $$x_1 x_2 x_3 + \Sigma a_i x_i^3,$$

(16) $$\Sigma a_i x_i^3 \qquad (a_1, a_2, a_3 \text{ not all zero}).$$

4. *No form in one of the six systems* (11)–(16) *is reducible to a form in another of the systems by a ternary linear transformation in the $GF[3^n]$.*

Indeed, let f_1, f_2, f_3 denote the partial derivatives of f, given by (1). The number N of sets of solutions x_1, x_2, x_3 in the $GF[3^n]$ of $f_1 = f_2 = f_3 = 0$ is invariant under linear transformation.* For (11)–(16), we have $N = 3^n$, 3^n, 3^{2n}, 1, $3^{n+1} - 2$, 3^{3n}, respectively. The only case needing comment is (14). Eliminating x_1 between $f_1 = 0$ and $f_2 = 0$, we get $x_1^3 - x_1 x_3^2 - s x_3^3 = 0$. But $t^3 - t - s = 0$ is irreducible since $\phi(s) \neq 0$. Hence $x_1 = x_3 = 0$, and then $x_2 = 0$ by $f_3 = 0$.

For (11) and (12), we have the same value of N. But for (12), Q is a not-square ν; while for (11), (13) and (16), $Q = 0$; for (14) and (15), $Q = 1$. Under a transformation of determinant D, Q becomes $D^2 Q$, so that the quadratic character of Q is an invariant.

5. In view of § 4 the problem of the reduction to canonical forms falls into six independent problems. Consider the forms $S(a_1, a_2, a_3)$ of any one of the six systems, and let T be a transformation of one of its forms $S(a_1', a_2', a_3')$ into a second. Since the modulus is 3, T transforms $\Sigma b_i x_i^3$ into a similar sum $\Sigma \beta_i x_i^3$. *A system of forms is invariant under every transformation which replaces one of its forms by a second.*

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* There exists an invariant (other than Q) involving b and the c_{ij} alone.

We proceed to determine the groups of transformations leaving invariant the various systems of forms. As the usual direct method would be laborious, special devices have been invented.

6. *Group of the system* (11). Suppose that a particular form

$$F'_{a'_1 a'_2 a'_3} \equiv x_1'^2 x_2' + x_2'^2 x_3' + \Sigma a_i' x_i'^3$$

becomes $F_{a_1 a_2 a_3}$, given by (11), in view of the transformation

$$(17) \qquad S:\ x_i' = \sum_{j=1}^{3} \alpha_{ij} x_j \qquad (i = 1, 2, 3),$$

with coefficients taken modulo 3. Then

$$\tfrac{1}{2}\frac{\partial^2 F}{\partial x_j \partial x_k} = x_1'(\alpha_{1j}\alpha_{2k} + \alpha_{1k}\alpha_{2j}) + x_2'(\alpha_{1j}\alpha_{1k} + x_{2j}\alpha_{3k} + \alpha_{2k}\alpha_{3j}) + x_3'\alpha_{2j}\alpha_{2k} \equiv \psi_{jk},$$

since $\partial x_i' / \partial x_j = \alpha_{ij}$. Hence

$$0 = \psi_{13} = \psi_{33},\ x_1 = \psi_{12},\ x_2 = \psi_{11} = \psi_{23},\ x_3 = \psi_{22}.$$

From $0 \equiv \psi_{33}$, we get $\alpha_{23} = \alpha_{13} = 0$. Hence $\alpha_{33} \neq 0$, since $|\alpha| \neq 0$. Then $\alpha_{21} = 0$ from $0 \equiv \psi_{13}$. Next, $\psi_{11} = \alpha_{11}^2 x_2$, $\psi_{23} = \alpha_{22}\alpha_{33} x_2$. Hence $\alpha_{11}^2 = \alpha_{22}\alpha_{33}$, and

$$S = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{11}^2 \alpha_{22}^{-1} \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} \alpha_{11}\alpha_{22} & \alpha_{11}\alpha_{12} + \alpha_{31}\alpha_{22} & 0 \\ 0 & \alpha_{11}^2 & 0 \\ -\alpha_{12}\alpha_{22} & \alpha_{12}^2 - \alpha_{22}\alpha_{32} & \alpha_{22}^2 \end{pmatrix}.$$

The necessary and sufficient conditions that $S^{-1}S = I$ are

$$\alpha_{11}^2 \alpha_{22} = 1, \quad \alpha_{31} = \alpha_{11}^3 \alpha_{12}.$$

Hence the transformations leaving system (11) invariant are

$$(18) \qquad x_1' = \alpha x_1 + \alpha_{12} x_2,\ x_2' = \alpha^{-2} x_2,\ x_3' = \alpha^3 \alpha_{12} x_1 + \alpha_{32} x_2 + \alpha^4 x_3 \qquad (\alpha \neq 0).$$

For the $GF[3^n]$, the order of the group is $3^{2n}(3^n - 1)$.

7. *Group of the system* (12). Evidently $(x_1^2 - \nu x_3^2) x_2$ is invariant under

$$(19) \qquad T_{\alpha,\beta}:\ x_1' = \alpha x_1 + \beta x_3,\ x_3' = \nu^{-1}\beta x_1 + \alpha x_3,\ x_2' = x_2/(\alpha^2 - \nu^{-1}\beta^2),$$

where α and β are any marks not both zero, and under

$$(20) \qquad L:\ x_3' = -x_3.$$

Let S, given by (17), be any transformation of the group of the system (12). We prove that S is generated by $T_{\alpha,\beta}$ and L. Now S and $ST_{0,1}$ do not both have $\alpha_{11} = \alpha_{13} \doteq 0$; let S' denote the one with α_{11} and α_{13} not both zero. Then, for suitable marks α, β, $T_{\alpha,\beta}^{-1} S' \equiv S_1$ replaces x_1 by $x_1 - t x_2$. From the conditions that S_1 shall replace $x_1^2 x_2 - \nu x_3^2 x_2$ by a form (12), we readily find that

S_1 is either the identity I or L. Now L transforms $T_{\alpha,\beta}$ into $T_{\alpha,-\beta}$. *Hence the group is composed of the* $2(3^{2n}-1)$ *transformations* $T_{\alpha,\beta}$ *and* $T_{\alpha,\beta}L$.

A simpler method of determining the group is to introduce the irrationality j, where $j^2=\nu$ defines the $GF[3^{2n}]$. Then $j^{3^n}=-j$. Introduce the conjugate variables

$$y=x_1+x_3j,\quad y^{3^n}=x_1-x_3j.$$

The form (12) becomes

$$(21)\qquad y\,y^{3^n}x_2+a_2x_2^3+Ay^3+(Ay^3)^{3^n},\quad A\equiv -a_1-a_3/j^3.$$

By inspection, or formally by § 10, this system of forms admits only the $2(3^{2n}-1)$ transformations T_t and T_tL, t any mark $\neq 0$ of the $GF[3^{2n}]$, where

$$(19')\qquad T_t:\quad y'=ty,\quad x_2'=t^{-3^n-1}x_2,$$

$$(20')\qquad L:\quad y'=y^{3^n}.$$

8. *Group of the system* (13). As in § 6, we get

$$\tfrac{1}{2}\frac{\partial^2F}{\partial x_j\,\partial x_k}=x_1'(\alpha_{1j}\alpha_{2k}+\alpha_{1k}\alpha_{2j})+x_2'\alpha_{1j}\alpha_{1k}.$$

For $j=k=2$, we get $\alpha_{12}=0$. For $k=3$, we get $\alpha_{1j}\alpha_{23}+\alpha_{13}\alpha_{2j}=0$, $\alpha_{1j}\alpha_{13}=0$, for $j=1, 2, 3$. For $j=3$, the latter gives $\alpha_{13}=0$. Then $\alpha_{1j}\alpha_{23}=0$ $(j=1, 2, 3)$. Hence $\alpha_{23}=0$. For $j=1$, $k=1$ and 2, we get

$$x_2=-x_1'\alpha_{11}\alpha_{21}+x_2'\alpha_{11}^2,\quad x_1=x_1'\alpha_{11}\alpha_{22}.$$

Then $SS^{-1}=I$ requires that $\alpha_{11}^2\alpha_{22}=1$. *System* (13) *is invariant under exactly the* $3^{3n}(3^n-1)^2$ *transformations*

$$(22)\quad x_1'=\alpha_{11}x_1,\ x_2'=\alpha_{21}x_1+\alpha_{11}^{-2}x_2,\ x_3'=\alpha_{31}x_1+\alpha_{32}x_2+\alpha_{33}x_3\ (\alpha_{11}\neq 0,\ \alpha_{33}\neq 0).$$

9. *Group of the system* (14). This case is the most difficult of all and requires a new device. We introduce a cubic irrationality i such that the "determinant" of the enlarged field $F(i)$ yields a ternary cubic form belonging to the system (14); the factorization of this determinant in $F(i)$ is known.* For (14), $\phi(s)\neq 0$, so that x^3-x-s is irreducible in the $GF[3^n]$. Hence

$$(23)\qquad i^3=i+s$$

defines the $GF[3^{3n}]$. To construct its determinant, expand †

$$(x_2+x_1i+x_3i^2)(y_2+y_1i+y_3i^2).$$

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* Dickson, *Transactions*, vol. 7 (1906), pp. 388, 389.

† The interchange of subscripts 1 and 2 is made here instead of later on.

The expansion may be exhibited with detached coefficients thus:

$$\begin{array}{c|lll} & y_1 & y_2 & y_3 \\ \hline 1 & s\,x_3 & x_2 & s\,x_1 \\ i & x_2 + x_3 & x_1 & x_1 + s\,x_3 \\ i^2 & x_1 & x_3 & x_2 + x_3 . \end{array}$$

The determinant is congruent modulo 3 to

$$(24) \qquad x_1^2 x_2 - x_3^2 x_2 + x_2^2 x_3 + s\,x_3^2 x_1 - s\,x_1^3 - x_2^3 - s^2 x_3^3 .$$

This form belongs to the system (14); indeed, it is the only form (14) differing from its Hessian (2) only by a constant factor (here -1).

In view of its origin, (24) has the factors

$$(25) \qquad \xi \equiv x_2 + x_1 i + x_3 i^2, \quad \xi_1 \equiv x_2 + x_1 i^{3^n} + x_3 i^{2 \cdot 3^n}, \quad \xi_2 \equiv x_2 + x_1 i^{3^{2n}} + x_3 i^{2 \cdot 3^{2n}}.$$

It follows from the theory of conjugate variables that (14) equals

$$(26) \qquad \xi \xi_1 \xi_2 + \beta \xi^3 + \beta^{3^n} \xi_1^3 + \beta^{3^{2n}} \xi_2^3,$$

or, as we may write,

$$(26') \qquad \xi^{1+3^n+3^{2n}} + \beta \xi^3 + (\beta \xi^3)^{3^n} + (\beta \xi^3)^{3^{2n}}.$$

In case we wish to pass from the forms and transformations in the $GF[3^{3n}]$ to those in the $GF[3^n]$, we need the value of β, viz.,

$$(27) \qquad \beta = -(a_1 + s)\, i^3 - (a_2 + 1)(i^4 - 1) - (a_3 + s^2).$$

On applying to (26′) the respective substitutions

$$(28) \qquad \xi' = t\,\xi,$$

$$(29) \qquad \xi' = t\,\xi^{3^n}, \qquad\qquad (t^{1+3^n+3^{2n}} = 1),$$

$$(30) \qquad \xi' = t\,\xi^{3^{2n}},$$

we obtain forms of type (26′) in which the new β's are

$$(31) \qquad \beta t^3, \quad \beta^{3^{2n}} t^{3^{2n+1}}, \quad \beta^{3^n} t^{3^{n+1}}.$$

Further, every automorph of the system (26), which preserves the conjugacy of the variables ξ, ξ_1, ξ_2, is given by (28), (29), or (30). *The group of the system* (14) *is of order* $3(1 + 3^n + 3^{2n})$; *it is generated by two operators U and V such that*

$$(32) \qquad U^{1+3^n+3^{2n}} = I, \quad V^3 = I, \quad V^{-1} U V = U^{3^n}.$$

10. *Group of the system* (15). An immediate application of the method of § 6 shows that the $6(3^n - 1)^2$ transformations are

$$(33) \qquad x_1' = \alpha x_i, \quad x_2' = \beta x_j, \quad x_3' = \alpha^{-1} \beta^{-1} x_k \qquad (i, j, k = 1, 2, 3).$$

11. The group of the system (16) is the general ternary group of order

$$\tau \equiv 3^{3n}(3^{3n}-1)(3^{2n}-1)(3^n-1). \tag{34}$$

12. As a check on the results in §§ 2–11, we note that

$$\frac{\tau\,3^{3n}}{3^{2n}(3^n-1)}+\frac{\tau\,3^{3n}}{2(3^{2n}-1)}+\frac{\tau\,3^{3n}}{3^{3n}(3^n-1)^2}+\frac{\tau\,3^{3n}}{3(1+3^n+3^{2n})}+\frac{\tau\,3^{3n}}{6(3^n-1)^2}+\frac{\tau(3^{3n}-1)}{\tau}$$

equals $3^{10n}-1$, the total number of forms (1), not identically zero.

13. *Reduction of forms* (11). The available transformations are given by (18), which replaces (11) by a similar form with

$$a_1' = a_1\alpha^3 + a_3\alpha^9\alpha_{12}^3,\quad a_2' = a_1\alpha_{12}^3 + a_2\alpha^{-6} + a_3\alpha_{32}^3 + \alpha^{-2}\alpha_{12}^2 + \alpha^{-4}\alpha_{32},\quad a_3' = a_3\alpha^{12}. \tag{35}$$

Let first $a_3 = 0$. Since every mark is a cube, we may take $a_1 = 0$ or 1. Then for $\alpha = 1$, $\alpha_{12} = 0$, $\alpha_{32} = -a_2$, we have $a_1' = 0$ or 1, $a_2 = 0$. We reach types 1 and 2 of the Table.

Let next $a_3 \neq 0$. We may take $a_1 = 0$. Then for $\alpha_{12} = 0$, $\alpha_{32} = \alpha^{-2}z$,

$$a_1' = 0,\quad a_3' = \alpha^{12}a_3,\quad a_2'\alpha^6 = a_2 + a_3z^3 + z. \tag{35$'$}$$

By choice of α and z, we can make $a_2' = 0$ or 1, if $n > 1$. Indeed, suppose that $\lambda \equiv a_2 + a_3z^3 + z$ is a not-square for every z in the $GF[3^n]$. Then

$$\lambda^{\frac{1}{2}(3^n-1)} \equiv \lambda\lambda^3\lambda^9\ldots\lambda^{3^{n-1}} = -1 \qquad \text{(for every } z\text{)}.$$

In the final factor, we replace z^{3^n} by z. Then the product is of degree

$$(3+9+27+\ldots+3^{n-1}) + 3^{n-1} \equiv \tfrac{1}{2}(5\,.\,3^{n-1}-3) < 3^n.$$

Hence the relation must be an identity in z. The coefficient of the highest power of z is a_3^q, $q = 1+3+\ldots 3^{n-2}$, if $n > 1$; but is a_3+1, if $n = 1$. Excluding for the present the case $n = 1$, we thus have $a_1 = 0$, $a_2 = 0$ or 1.

For $a_1 = a_2 = 0$, $a_3 \neq 0$, the only further normalization is the specialization of a_3' by the choice of α^{12}. Let ρ be a primitive root of the $GF[3^n]$. If n is odd, the even powers of ρ are 12th powers, since

$$\rho^2 = \rho^2\rho^{3^n-1} = \{\rho^{\frac{1}{4}(3^n+1)}\}^4,\quad \tau = \{\tau^{3^{n-1}}\}^3;$$

while the odd powers of ρ are not-squares and hence not 12th powers. Hence for n odd, we may set $a_3' = \pm 1$, and reach types 3 and 4 of the Table. If n is even, the greatest common divisor of 12 and 3^n-1 is 4, so* that the only 12th powers are ρ^4, ρ^8, $\ldots$, $\rho^{3^n-1} \equiv 1$. Hence we may set $a_3' = 1$, ρ, ρ^2, or ρ^3, and reach types 5 and 6 of the Table.

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* *Linear Groups* (Teubner, 1901), p. 45, § 63.

Next, let $a_1 = 0$, $a_2 = 1$, $a_3 = -a \neq 0$. We first seek the conditions under which this case can be reduced to the preceding. Such a reduction occurs if and only if

$$0 = 1 - a z^3 + z \equiv Z \tag{36}$$

is solvable for z in the $GF[3^n]$. Now

$$Z + a Z^3 + a^4 Z^9 + \dots + a^{1+3+3^2+\dots+3^{n-2}} Z^{3^{n-1}} = \psi(a) + z - a^{\frac{1}{2}(3^n-1)} z^{3^n} \tag{37}$$

where

$$\psi(a) \equiv 1 + a + a^4 + a^{13} + \dots + a^{1+3+3^2+\dots+3^{n-2}}. \tag{38}$$

Hence the equation $0 = Z$ is solvable for z in the $GF[3^n]$ only in the following cases: either a is a not-square, or a is a square such that $\psi(a) = 0$.

Let therefore a be a square such that $\psi(a) \neq 0$. We discuss the values taken by $a' = Z^2 a$, when z is chosen in the $GF[3^n]$ so that $Z \equiv 1 - a z^3 + z$ is a 6th power ($Z = \alpha^6$). Set $\psi = \psi(a)$, $\psi' = \psi(a')$. In view of (38), with a replaced by a', we get

$$Z\psi' = Z + a Z^3 + a^4 Z^9 + \dots + a^{1+3+\dots+3^{n-2}} Z^{3^{n-1}}. \tag{39}$$

Since a is a square and $z^{3^n} = z$, we deduce $Z\psi' = \psi$ from (37) and (39). To make Z a 6th power, it suffices to make it a square. Hence a necessary condition for the equivalence of two forms with the parameters a and a' is that ψ/ψ' be a square. We next show that this condition is sufficient. Let a and a' be given squares such that ψ/ψ' is a square. Now

$$\psi - a\psi^3 = 1 - a^{\frac{1}{2}(3^n-1)} = 0.$$

Hence $1 = a\psi^2$, $1 = a'\psi'^2$. Hence the equation $a' = Z^2 a$ is satisfied by the *square* $Z = \psi/\psi'$. For this value of Z, (37) and (39) give

$$0 = z - z^{3^n},$$

so that $Z = 1 - a z^3 + z$ is solvable for z in the $GF[3^n]$. *For the forms* (11) *in which* $a_1 = 0$, $a_2 = 1$, $a_3 = -a$, *a being a square* $\neq 0$ *such that* $\psi(a) \neq 0$, *those with* $\psi(a)$ *a square are equivalent, likewise those with* $\psi(a)$ *a not-square, while the two sets are not equivalent.* The two resulting canonical forms are listed under 7 in the Table.

It remains to treat the case $n = 1$, $a_3 \neq 0$, $a_1 = 0$. By (35), for $\alpha_{12} = 0$, $a'_1 = 0$, $a'_2 = a_2 + (a_3 + 1)\alpha_{32}$, $a'_3 = a_3$. If $a_3 = 1$, we can make $a'_2 = 0$, and reach type 3 of the Table. If $a_3 = -1$, no normalization is possible, and we have types 4 and 8 of the Table.

The determination of the automorphs of types 1–8 follows readily from (35).

Comment is needed only in the case of 7; viz., $a_1 = 0$, $a_2 = 1$, $a_3 = -a \neq 0$. Then

$$\alpha_{12} = 0, \ \alpha^4 = 1, \ 1 = \alpha^2 - a\,\alpha_{32}^3 + \alpha_{32}.$$

The case $\alpha^2 = 1$ leads to the automorphs listed. The case $\alpha^2 = -1$ is excluded, since $z \equiv \alpha_{32}$ would then be a root of (36), whereas the latter was shown to be not solvable when a is subject to the conditions on type 7.

14. *Reduction of forms* (12). The available transformations are $T_{\alpha,\beta}$ and $T_{\alpha,\beta}\,L$, given by (19) and (20). Now $T_{\alpha,\beta}$ replaces (12) by a similar form with

$$a_1' = a_1\alpha^3 + a_3\beta^3\nu^{-3}, \quad a_3' = a_1\beta^3 + a_3\alpha^3, \quad a_2' = a_2/(\alpha^2 - \nu^{-1}\beta^2)^3.$$

If a_1 and a_3 are not both zero, we can determine α and β to make $a_1' = 1$, $a_3' = 0$; the resulting type is 11 of the Table. If $a_1 = a_3 = 0$, we can make $a_2' = 0$ or 1; the resulting types are 9 and 10 of the Table. Since L leaves unaltered types 9, 10, 11, no two of them are equivalent.

15. *Reduction of forms* (13). Applying (22), we get

$$a_1' = \alpha_{11}^2\alpha_{21} + a_1\alpha_{11}^3 + a_2\alpha_{21}^3 + a_3\alpha_{31}^3, \quad a_2' = a_2\alpha_{11}^{-6} + a_3\alpha_{32}^3, \quad a_3' = a_3\alpha_{33}^3.$$

For $a_3 \neq 0$ we reach type 14 of the Table. For $a_3 = 0$, we first make $a_2 = 0$, -1, or $-\nu$, where ν is a particular not-square. For $a_3 = a_2 = 0$, we take $\alpha_{11} = 1$, $\alpha_{21} = -a_1$, and reach type 12. For $a_3 = 0$, $a_2 = -1$, we take $\alpha_{11} = \pm 1$ to preserve $a_2' = -1$. The problem of the specialization of $a_1' \equiv \pm a_1 + \alpha_{21} - \alpha_{21}^3$ is essentially the same problem as the specialization of s in (14). As at the end of § 2, we may take a_1' to be zero or a particular solution of $\phi(c) = 1$. The resulting types are 15 and 16 of the Tables.

Finally, for $a_3 = 0$, $a_2 = -\nu$, we take $\alpha_{11} = \pm 1$ to preserve $a_2' = -\nu$ Setting $\alpha_{21} = \pm z$, we have $\pm a_1' = a_1 + z - \nu z^3 \equiv W$. Then

$$W + \nu W^3 + \nu^4 W^9 + \dots + \nu^{1+3+\dots+3^{n-2}} W^{3^{n-1}} = \alpha + z - \nu^{\frac{1}{2}(3^n-1)} z^{3^n},$$

$$\alpha \equiv a_1 + \nu a_1^3 + \nu^4 a_1^9 + \dots + \nu^{1+3+\dots+3^{n-2}} a_1^{3^{n-1}}.$$

Since $\nu^{\frac{1}{2}(3^n-1)} = -1$, the equation $W = 0$ has the root* $z = \alpha$. Hence we may set $a_1' = 0$ and have type 13 of the Table.

16. The reduction of the forms (14)–(16) to the types 17–23 of the Tables offers no difficulty in view of the results of §§ 9–11.

17. In addition to the check in § 12 on the six systems, we note that the canonical forms within each system and their automorphs were checked by making two counts of the forms of the system.

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* The existence of a root also follows from a theorem on the analytic representation of substitutions, *Linear Groups*, § 31, p. 57.

	Canonical form	Automorphs	Number of automorphs
1	$A \equiv x_1^2 x_2 + x_2^2 x_3$	$(a x_1 + \beta x_2,\ a^{-2} x_2,\ a^3 \beta x_1 - a^2 \beta^2 x_2 + a^4 x_3)$	$3^n (3^n - 1)$
2	$A + x_1^3$	$(x_1 + \beta x_2,\ x_2,\ \beta x_1 - (\beta^3 + \beta^2) x_2 + x_3)$	3^n
3	$A + x_3^3$ (n odd)	$(\pm x_1,\ x_2,\ x_3)$	2
4	$A - x_3^3$ (n odd)	$(\pm x_1,\ x_2,\ \beta x_2 + x_3)$, $\beta^3 = \beta$	6
5	$A + \rho^i x_3^3$ (n even, $i = 0, 2$)	$(a x_1,\ a^2 x_2,\ \beta x_2 + x_3)$, $a^4 = 1$, $\rho^i \beta^3 = -\beta$	12
6	$A + \rho^i x_3^3$ (n even, $i = 1, 3$)	$(a x_1,\ a^2 x_2,\ x_3)$, $a^4 = 1$	4
7	$A + x_2^3 - a x_3^3$ ($n > 1$)	$(\pm x_1,\ x_2,\ \beta x_2 + x_3)$, $a \beta^3 = \beta$	6
8	$A \pm x_2^3 - x_3^3$ ($n = 1$)	$(\pm x_1,\ x_2,\ \beta x_2 + x_3)$	6
9	$B \equiv x_1^2 x_2 - \nu x_3^2 x_2$	$T_{\alpha,\beta}$, $T_{\alpha,\beta} L$ (§ 7)	$2 (3^{2n} - 1)$
10	$B + x_2^3$	$T_{\alpha,\beta}$, $T_{\alpha,\beta} L$, $\alpha^2 - \nu^{-1} \beta^2 = 1$	$2 (3^n + 1)$
11	$B + x_1^3 + a_2 x_2^3$	$(x_1,\ x_2,\ \pm x_3)$	2
12	$x_1^2 x_2$	(22) with $a_{21} = 0$	$3^{2n} (3^n - 1)^2$
13	$x_1^2 x_2 - \nu x_2^3$	$(\pm x_1,\ x_2,\ \beta x_1 + \gamma x_2 + a x_3)$	$2 . 3^{2n} (3^n - 1)$
14	$x_1^2 x_2 + x_3^3$	$(a x_1,\ a^{-2} x_2 - a^{-2} \beta^3 x_1,\ x_3 + \beta x_1)$	$3^n (3^n - 1)$
15	$x_1^2 x_2 - x_2^3$	$(\pm x_1,\ \beta x_1 + x_2,\ \gamma x_1 + \delta x_2 + \varepsilon x_3)$, $\beta^3 = \beta$	$6 . 3^{2n} (3^n - 1)$
16	$x_1^2 x_2 - x_2^3 + a_1 x_1^3$	$(x_1,\ \beta x_1 + x_2,\ \gamma x_1 + \delta x_2 + \varepsilon x_3)$, $\beta^3 = \beta$	$3 . 3^{2n} (3^n - 1)$
17	$(24) \sim (26)_{\beta \equiv 0}$	(28)-(30)	$3 (1 + 3^n + 3^{2n})$
18	$(14) \sim (26)_{\beta \neq 0}$	I, (29) with $t = \beta^{3^{n-1} - 3^{3n-1}}$ (30) with $t = \beta^{3^{2n-1} - 3^{3n-1}}$	3
19	$x_1 x_2 x_3$	(33)	$6 (3^n - 1)^2$
20	$x_1 x_2 x_3 + x_1^3$	$(x_1,\ a x_i,\ a^{-1} x_j)$, $i, j = 2, 3$	$2 (3^n - 1)$
21	$x_1 x_2 x_3 + x_1^3 + x_2^3$	I, $(x_2,\ x_1,\ x_3)$	2
22	$x_1 x_2 x_3 + a \Sigma x_i^3$	Permutations of x_1, x_2, x_3	6
23	x_1^3	$(x_1,\ \Sigma a_j x_j,\ \Sigma \beta_j x_j)$	$3^{3n} (3^{2n} - 1) (3^n - 1)$

Specification of the parameters in the canonical forms.

5, 6	ρ is a fixed primitive root of the $GF[3^n]$.
7	a has two values, each a square in the $GF[3^n]$. For one, $\psi \equiv 1 + a + a^4 + a^{13} + \ldots + a^{1+3+3^2+\ldots+3^{n-2}}$ is a square; for the other, ψ is a not-square.
9, 13	ν is a fixed not-square in the $GF[3^n]$.
11	a_2 ranges over the 3^n marks of the $GF[3^n]$.
16	a_1 is a particular solution of $\phi(c) \equiv c + c^3 + \ldots + c^{3^{n-1}} = 1$.
18	β ranges over the $3^n - 1$ multipliers in a rectangular table of the marks $\neq 0$ of the $GF[3^{3n}]$, those in the first row being the roots of $t^{1+3^n+3^{2n}} = 1$.
22	α ranges over the $3^n - 1$ marks $\neq 0$ of the $GF[3^n]$.

Thus for system (12), with the canonical forms 9–11,

$$\frac{2(3^{2n}-1)}{2(3^{2n}-1)}+\frac{2(3^{2n}-1)}{2(3^n+1)}+\frac{2(3^{2n}-1)}{2}\cdot 3^n=3^{3n}.$$

18. Aside from 17 and 18, all the canonical forms and automorphs in the Table have their coefficients in the initial field $GF[3^n]$. A similar treatment of 17 and 18 for $n=1$ will be given for illustration. We may set $s=1$; then the roots of (23) belong to the exponent 13. As the three canonical forms, we may take (14) with $s=1$ and

(40) $$(a_1, a_2, a_3)=(-1, -1, -1),\ (0, 0, 0),\ (0, 0, -1).$$

Indeed, for these, β given by (27) equals 0, $-i^2+1$, $-i^2-1$, respectively; while $(-i^2-1)/(-i^2+1)=1-i$ belongs to the exponent 26, so that no two of the forms given by (40) are equivalent. For $t=i$, (28) gives

(41) $$U:\ x_1'=x_2+x_3,\ x_2'=x_3,\ x_3'=x_1 \qquad (U^{13}=I).$$

For $\beta=1-i^2$, we get $\beta^{18}=-1-i^2$. Hence (29), for $t=-1-i^2$, gives

(42) $$V:\ x_1'=x_1+x_3,\ x_2'=x_1-x_2,\ x_3'=-x_1-x_2 \qquad (V^3=I),$$

an automorph of $x_1^2x_2-x_3^2x_2+x_2^2x_3+x_3^2x_1$. Hence (41) and (42) generate the group (32) of the system (14); its 39 transformations are the automorphs of (40_1). The 3 automorphs of (40_2) are the powers of V. The 3 automorphs of (40_3) are obtained similarly.

In (26′) for $n=1$, set $\xi=\beta^4y$. For β a square, $\beta^{13}=1$, we find that the form vanishes for 13 values of y, since $y^{12}+y^8+y^2+1$ divides $y^{26}-1$, modulo 3. For $\beta^{13}=-1$, the form vanishes for 7 values of y, since $-y^{12}+y^8+y^2+1$ and $y^{26}-1$ have the greatest common divisor y^6-y^2-1. Hence the forms (40) vanish for exactly 1, 7 and 13 respective sets of values x_i, modulo 3, since $1-i^2$ is a not-square and $-1-i^2$ a square in the $GF[3^3]$.

As the basis for a similar treatment for $n>1$, we note that, at least when $n=1, 2, 3, 4$, a root i of (23) is a primitive root in the $GF[3^{3n}]$ when s is a primitive root in the $GF[3^n]$.

19. Instead of forms 9, 10, 11, we may, on the basis of § 7, employ

Canonical form	Automorphs	Number
$y\,y^{3^n}x_2$	T_t, T_tL, defined by (19′), (20′)	$2(3^{2n}-1)$
$y\,y^{3^n}x_2+x_2^3$	Preceding with $t^{3^n+1}=1$	$2(3^n+1)$
$y\,y^{3^n}x_2+y^3+y^{3^{n+1}}+a_2x_2^3$	I, L	2

Hence under transformations with coefficients in a higher field, types 9–11, 17, 18 reduce to types analogous to 19–22.

20. Theorem. *Every ternary cubic form in the* $GF[3^n]$ *can be reduced, by a ternary linear transformation rational or irrational with respect to the given field, to one and but one of the eleven ultimate canonical forms* 1, 2, 3, 12, 14, 15, 19–23.

For systems (12), (14) and (15), the invariant Q is not zero. In view of § 19, the ultimate canonical forms are 19–22. For the remaining systems (11), (13) and (16), $Q=0$, so that reduction to 19–22 is impossible; while (16) alone is reducible to 23. As in §§ 6, 8, no form (13) is reducible to a form (11) by a transformation of modulus 3. In the reduction of forms (11), in § 13, the case $a_3=0$ led to types 1, 2. For $a_3 \neq 0$, conditions (35′) may be satisfied for α and z in a higher field, when we set $a_3'=1$, $a_2'=0$. After the determination of α^3 as a fourth root of a_3^{-1}, α^3 follows rationally; the determination of z requires the solution of a cubic. Hence (11) with $a_3 \neq 0$ can be reduced to type 3 by a transformation in the $GF[3^{12n}]$.

In the reduction of forms (13) in § 15, the case $a_3 \neq 0$ led to type 14, the case $a_3=a_2=0$ to type 12. For $a_3=0$, $a_2 \neq 0$, we may make $a_2=-1$, $a_1=0$ by using a quadratic and a cubic irrationality; thus in the $GF[3^{6n}]$ we reach type 15.

The present list of ultimate forms for modulus 3 differs from Gordan's list of non-modular forms. His C_2, C_8, C_{10} are here equivalent to 23; C_1 to 22, C_3 to 21, C_4 to 20, C_5 to 19, C_6 to 14, C_7 to 1, C_9 to 12, while none correspond to 2, 3, 15.

The University of Chicago, *September*, 1906.

The Elliptic Cylinder Function of Class K.

By William H. Butts.

The object of this paper is the synthetic treatment of the Elliptic Function of Class K and the computation of tables of values that may be useful to physicists in studying the properties of elliptical membranes and elliptical cylinders. The literature on this subject is very limited, as the only serious attempts to discuss this function have been made by Mathieu and Heine. The investigation of Mathieu in his Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique (*Jour. de Liouville*) made some progress in the theoretical treatment of the function, but did not lead to definite conclusions. Heinrich Weber, in *Annal. von Clebsch und Neumann*, Bd. I, says of Mathieu's work: "Die Integration ist dort durch Reihen bewerkstelligt, von denen mit grossem Fleisse eine beträchtliche Anzahl Glieder berechnet sind, für welche aber ebenfalls kein allgemeines Gesetz angegeben ist. Diese Untersuchungen mögen daher für den Physiker immerhin von grossem Werte sein, mathematisch scheint mir das Problem dadurch der Lösung wenig näher gebracht zu sein, als durch die Aufstellung der gewöhnlichen Differentialgleichung selbst."

E. Heine, in *Kugelfunktionen*, Bd. I, II, makes greater progress in the analytic treatment, but does not give a satisfactory proof[*] of the convergence of the series and does not carry the investigation far enough to make the results useful to the physicist.

Laplace's equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ is reduced by Heine, after various transformations and the introduction of Lamé's elliptical coordinates, to the convenient form

$$(1) \qquad \frac{d^2 E}{d\phi^2} + \left(\frac{8}{b} \cos 2\phi + 4z\right) E = 0.$$

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[*] Since the above statement was written, a satisfactory proof has appeared in the Inaugural Dissertation of Simon Dannacher, Zürich, 1906.

Our investigation will be limited to the function of the Class K,

$$E = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi, \tag{2}$$

in which a is independent of ϕ but a function of z and of the argument b. Under what conditions is this a solution of (1)?

Substituting this value of E in (1), we find the following recursion formulas, showing the relation between the coefficients:

$$\begin{aligned} a_1 &= -\tfrac{1}{2}bza_0, \\ a_2 &= b(1-z)a_1 - a_0, \\ a_3 &= b(4-z)a_2 - a_1, \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ a_{n+1} &= b(n^2-z)a_n - a_{n-1}. \end{aligned} \tag{3}$$

To determine the necessary conditions for the convergence of (2), it is necessary first to show that $\lim_{n=\infty} a_n = 0$.

Substituting a finite portion of (2) in (1) gives

$$\frac{d^2E}{d\phi^2} + \left(\frac{8}{b}\cos 2\phi + 4z\right)E = \frac{4}{b}\left(a_n \cos(2n+2)\phi - a_{n-1}\cos 2n\phi\right). \tag{4}$$

If values of z can be found that will make a_n and a_{n-1} arbitrarily small, the first member may be made to approximate zero.

Assuming that $a_0 = 1$, a_1, a_2, ..., a_n are rational integral functions of z, and equations (3), in the form $a_n = f(z) = 0$, can be solved for various values of the argument b. Equations (3) give the following values of a_1, a_2, ..., a_7, explicitly in terms of b and z. Placing these equal to zero we compute the values of z that will make a_1, ..., a_7 approximately zero.

$$\begin{aligned} a_1 &= -\frac{1}{2}bz = 0, \\ a_2 &= +\frac{1}{2}b^2\left(z^2 - z - \frac{2}{b^2}\right) = 0, \\ a_3 &= -\frac{1}{2}b^3\left(z^3 - 5z^2 + \left(4 - \frac{3}{b^2}\right)z + \frac{8}{b^2}\right) = 0, \\ a_4 &= +\frac{1}{2}b^4\left(z^4 - 14z^3 + \left(49 - \frac{4}{b^2}\right)z^2 + \left(-36 + \frac{36}{b^2}\right)z + \left(-\frac{72}{b^2} + \frac{2}{b^4}\right)\right) = 0, \end{aligned} \tag{5}$$

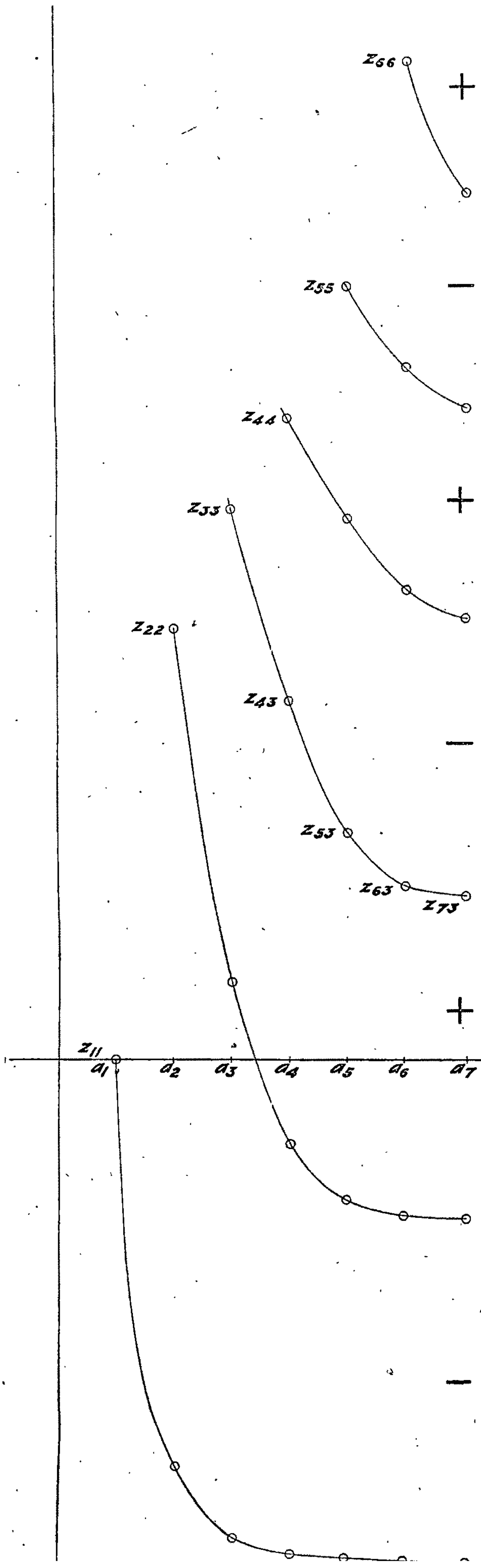

FIG. 1

$a_n = 0$

$n = 1, 2, --. 7$

$b = 0.1$

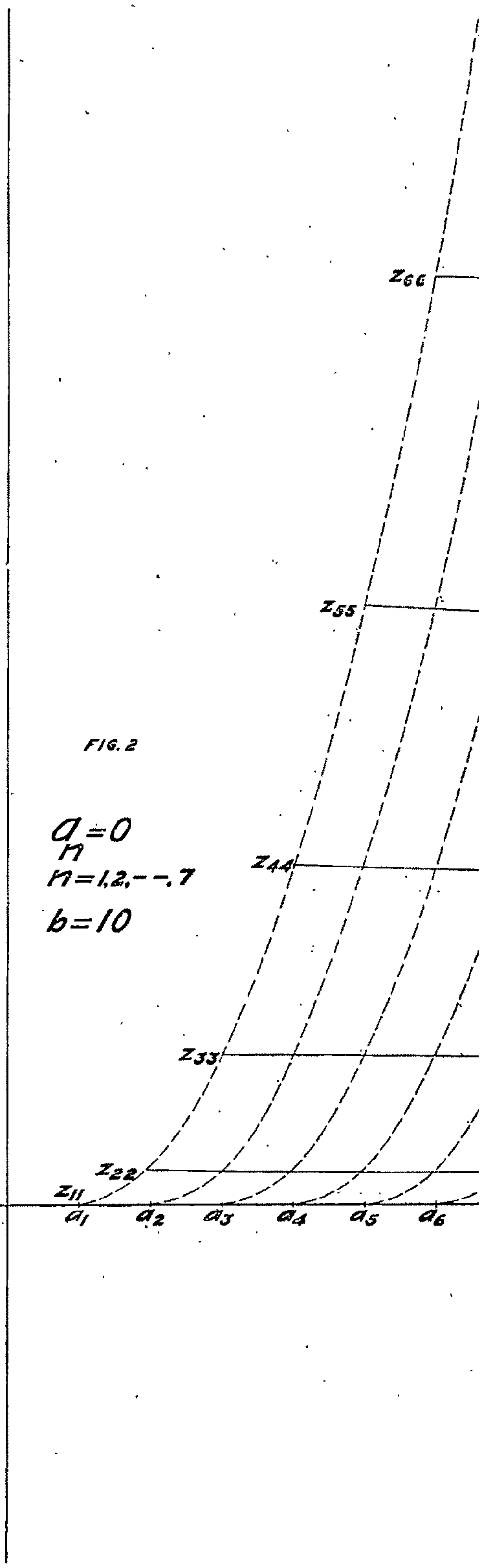

FIG. 2

$a_n = 0$

$n = 1, 2, --. 7$

$b = 10$

$$a_5 = -\frac{1}{2}b^5\Big(z^5 - 30z^4 + \Big(273 - \frac{5}{b^2}\Big)z^3 + \Big(-820 + \frac{105}{b^2}\Big)z^2$$
$$+ \Big(576 - \frac{652}{b^2} + \frac{5}{b^4}\Big)z + \Big(\frac{1152}{b^2} - \frac{40}{b^4}\Big)\Big) = 0,$$

$$a_6 = +\frac{1}{2}b^6\Big(z^6 - 55z^5 + \Big(1023 - \frac{6}{b^2}\Big)z^4 + \Big(-7645 + \frac{244}{b^2}\Big)z^3$$
$$+ \Big(21076 - \frac{3326}{b^2} + \frac{9}{b^4}\Big)z^2 + \Big(-14400 + \frac{17488}{b^2} - \frac{201}{b^4}\Big)z$$
$$+ \Big(-\frac{28800}{b^2} + \frac{1072}{b^4} - \frac{2}{b^6}\Big)\Big) = 0,$$

$$a_7 = -\frac{1}{2}b^7\Big(z^7 - 91z^6 + \Big(3003 - \frac{7}{b^2}\Big)z^5 + \Big(-44473 + \frac{490}{b^2}\Big)z^4$$
$$+ \Big(296296 - \frac{12383}{b^2} + \frac{14}{b^4}\Big)z^3 + \Big(-773136 + \frac{138044}{b^2} - \frac{630}{b^4}\Big)z^2$$
$$+ \Big(578400 - \frac{658944}{b^2} + \frac{8960}{b^4} - \frac{7}{b^6}\Big)z + \Big(\frac{1036800}{b^2} - \frac{39744}{b^4} + \frac{112}{b^6}\Big)\Big) = 0.$$

In order to study these equations and to deduce from them properties of the equation $a_\infty = 0$, we compute tables of values showing the roots to four decimal places and the values of the functions $a_1, a_2, \ldots, a_7$ for these approximate roots. The Tables I, II, III are given for $b = 0.1$, $b = 10$ and $b = 100$. The last figure of every result is given exactly as found, and is not increased by unity if the next figure is five or more.

Tables I, II, III and Figs. 1, 2 show how rapidly these roots approach a constant limiting value as n increases without limit. This line of argument forces us to use a_∞ as a function of z of infinite degree and to treat $a_\infty = 0$ as an equation of infinite degree. This extension of the function concept may be justified by the same necessity which forces us in certain problems to use infinity as a limit.

From equations (5) we deduce the following:

$$b = 0.1.$$

$$(6)\quad a_1 = -\frac{0.1}{2}(z) = 0,$$

$$a_2 = +\frac{(0.1)^2}{2}(z^2 - z - 200) = 0,$$

$$a_3 = -\frac{(0.1)^3}{2}(z^3 - 5z^2 - 296z + 800) = 0,$$

$$a_4 = +\frac{(0.1)^4}{2}(z^4 - 14z^3 - 351z^2 + 3564z + 12800) = 0,$$

$$a_5 = -\frac{(0.1)^5}{2}(z^5 - 30z^4 - 227z^3 + 9680z^2 - 14624z - 284800) = 0,$$

$$a_6 = +\frac{(0.1)^6}{2}(z^6 - 55z^5 + 423z^4 + 16755z^3 - 221524z^2 - 275600z + 5840000) = 0,$$

$$a_7 = -\frac{(0.1)^7}{2}(z^7 - 91z^6 + 2303z^5 + 4527z^4 - 802004z^3 + 6731264z^2 + 17224000z - 181760000) = 0.$$

$$b = 10.$$

$$(7)\quad a_1 = -\frac{10}{2}(z) = 0,$$

$$a_2 = +\frac{10^2}{2}(z^2 - z - 0.02) = 0,$$

$$a_3 = -\frac{10^3}{2}(z^3 - 5z^2 + 3.97z + 0.08) = 0,$$

$$a_4 = +\frac{10^4}{2}(z^4 - 14z^3 + 48.96z^2 - 35.64z - 0.7198) = 0,$$

$$a_5 = -\frac{10^5}{2}(z^5 - 30z^4 + 272.95z^3 - 818.95z^2 + 569.4805z + 11.516) = 0,$$

$$a_6 = +\frac{10^6}{2}(z^6 - 55z^5 + 1022.94z^4 - 7642.56z^3 + 21042.7409z^2 - 14225.1401z - 287.892802) = 0,$$

$$a_7 = -\frac{10^7}{2}(z^7 - 91z^6 + 3002.93z^5 - 44468.1z^4 + 296172.1714z^3 - 771755.623z^2 + 511811.455993z + 10364.025712) = 0.$$

$$b = 100.$$

$$(8)\quad a_1 = -\frac{10^2}{2}(z) = 0,$$

$$a_2 = +\frac{10^4}{2}(z^2 - z - 0.0002) = 0,$$

$$a_3 = -\frac{10^6}{2}(z^3 - 5z^2 + 3.9997z + 0.0008) = 0,$$

$$a_4 = +\frac{10^8}{2}(z^4 - 14z^3 + 48.9996z^2 - 35.9964z - 0.00719998) = 0,$$

$$a_5 = -\frac{10^{10}}{2}(z^5 - 30z^4 + 272.9995z^3 - 819.9895z^2 + 575.93480005z + 0.1151996) = 0,$$

$$a_6 = +\frac{10^{12}}{2}(z^6 - 55z^5 + 1022.9994z^4 - 7644.9756z^3 + 21075.66740009z^2 - 14398.25120201z - 2.379989280002) = 0,$$

$$a_7 = -\frac{10^{14}}{2}(z^7 - 91z^6 + 3002.9993z^5 - 44472.951z^4 + 296294.76170014z^3 - 773122.1956063z^2 + 518334.105689599993z + 103.679602560112) = 0.$$

These equations (6), (7), (8) are solved by Horner's method, and the last remainders are multiplied by the factors before the parentheses in the equations under consideration. By z_{63} we represent the third root of $a_6 = 0$, and by a_{63} the value of a_6 for this approximate z_{63}. See Tables I, II, III.

MAXIMUM AND MINIMUM VALUES OF a_∞.

The maxima and minima values of a_n can not be found by the usual methods when n increases beyond all limits; since a_∞ can not be expressed explicitly in terms of z, and $a'_\infty = 0$ is an equation of an infinite degree.

Substituting $a_n = b^n(n-1)!^2\ \beta_n$,* we compute the values of $\bar{z}$ in $\beta'_n = 0$ for finite values of n, $\bar{z}_{ni}$ being taken graphically as the abscissa corresponding to the maximum or minimum value of β_n between the i^{th} and $i+1^{th}$ roots of $\beta_n = 0$. We compute $\bar{z}$ from $\beta'_n = 0$ by Horner's method, accurate to one decimal place. Substituting these values of $\bar{z}$ in β_n by Horner's method, we obtain close approximations for $\overline{\beta}_n$.

I. *Argument* $b = 10$.

(9) $$\beta_n = \frac{a_n}{b^n(n-1)!^2}.$$

(10) $$\beta_2 = +\frac{1}{2}(z^2 - z - 0.02),$$

$$\beta_3 = -\frac{1}{8}(z^3 - 5z^2 + 3.97z + 0.08),$$

$$\beta_4 = +\frac{1}{72}(z^4 - 14z^3 + 48.96z^2 - 35.64z - 0.7198),$$

$$\beta_5 = -\frac{1}{1152}(z^5 - 30z^4 + 272.95z^3 - 818.95z^2 + 569.4805z + 11.516),$$

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*See Dannacher, p. 16.

$$\beta_6 = +\frac{1}{28800}(z^6 - 55z^5 + 1022.94z^4 - 7642.56z^3 + 21042.7409z^2 - 14225.1401z - 287.892802),$$

$$\beta_7 = -\frac{1}{103600}(z^7 - 91z^6 + 3002.93z^5 - 44468.1z^4 + 296172.1714z^3 - 771755.623z^2 + 511811.455993z + 10364.025712).$$

From these equations Table V is computed.

Substituting (9) in the recursion formula,

$$a_n = b\left((n-1)^2 - z\right) a_{n-1} - a_{n-2}, \tag{11}$$

$$b^n (n-1)!^2\, \beta_n = b\left((n-1)^2 - z\right) b^{n-1} (n-2)!^2 - b^{n-2} (n-3)!^2\, \beta_{n-2},$$

$$\beta_n = \left(1 - \frac{z}{(n-1)^2}\right) \beta_{n-1} - \frac{1}{b^2 (n-1)^2 (n-2)^2}\, \beta_{n-2}. \tag{12}$$

The first minima for β_n are found by computation to have the relation $|\overline{\beta}_{21}| > |\overline{\beta}_{31}| > |\overline{\beta}_{41}|$, and all are negative. See Table V.

$$\overline{\beta}_{31} = \left(1 - \frac{0.4}{2^2}\right) \beta_{21} - \frac{1}{10^2 \times 2^2 \times 1^2}\, \beta_{11}, \text{ for } \bar{z}_{31} = +0.4. \tag{13}$$

$$\overline{\beta}_{41} = \left(1 - \frac{0.4}{3^2}\right) \beta_{31} - \frac{1}{10^2 \times 3^2 \times 2^2}\, \beta_{21}, \text{ for } \bar{z}_{41} = +0.4. \tag{14}$$

$$\overline{\beta}_{51} = \left(1 - \frac{0.4}{4^2}\right) \beta_{41} - \frac{1}{10^2 \times 4^2 \times 3^2}\, \beta_{31}, \text{ for } \bar{z}_{41} = +0.4. \tag{15}$$

$$\overline{\beta}_{\infty 1} = \left(1 - \frac{\bar{z}_{\infty 1}}{\infty}\right) \beta_{\infty -1, 1} - \frac{1}{10^2 \times \infty^2 \times \infty^2}\, \beta_{\infty -2, 1}, \text{ for } \bar{z}_{\infty 1}. \tag{16}$$

The first two roots of $\beta_n = 0$ must lie between -2 and 0, $+1$ and $+4$, respectively, as proved by Heine.* Hence the first minimum lies between -2 and $+4$. Beginning with $\overline{\beta}_{51}$, the parenthesis of (15) cannot vary from unity by more than one-fourth, and the parenthesis rapidly approaches the limit unity in $\overline{\beta}_{n1}$ as n increases without limit. The last term in (15) equals $-0.0007\,\beta_{31}$ and in subsequent equations rapidly vanishes, so that the first term in (15) and in subsequent equations controls the sign and $\overline{\beta}_{n1}$ approaches the limit $\overline{\beta}_{n-1,1}$ for $\bar{z}_{n1}$. Hence $|\overline{\beta}_{n1}| < |\overline{\beta}_{n-1,1}| \cdots < |\overline{\beta}_{51}|$ and

$$\lim_{n=\infty} \overline{\beta}_{n1} = \overline{\beta}_{n-1,1}. \tag{17}$$

From computations it is evident that the first minimum of $\beta_\infty = 0$ is numerically less than 0.11 and negative; approximately, -0.09.

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* *Kugelfunktionen*, I, 412.

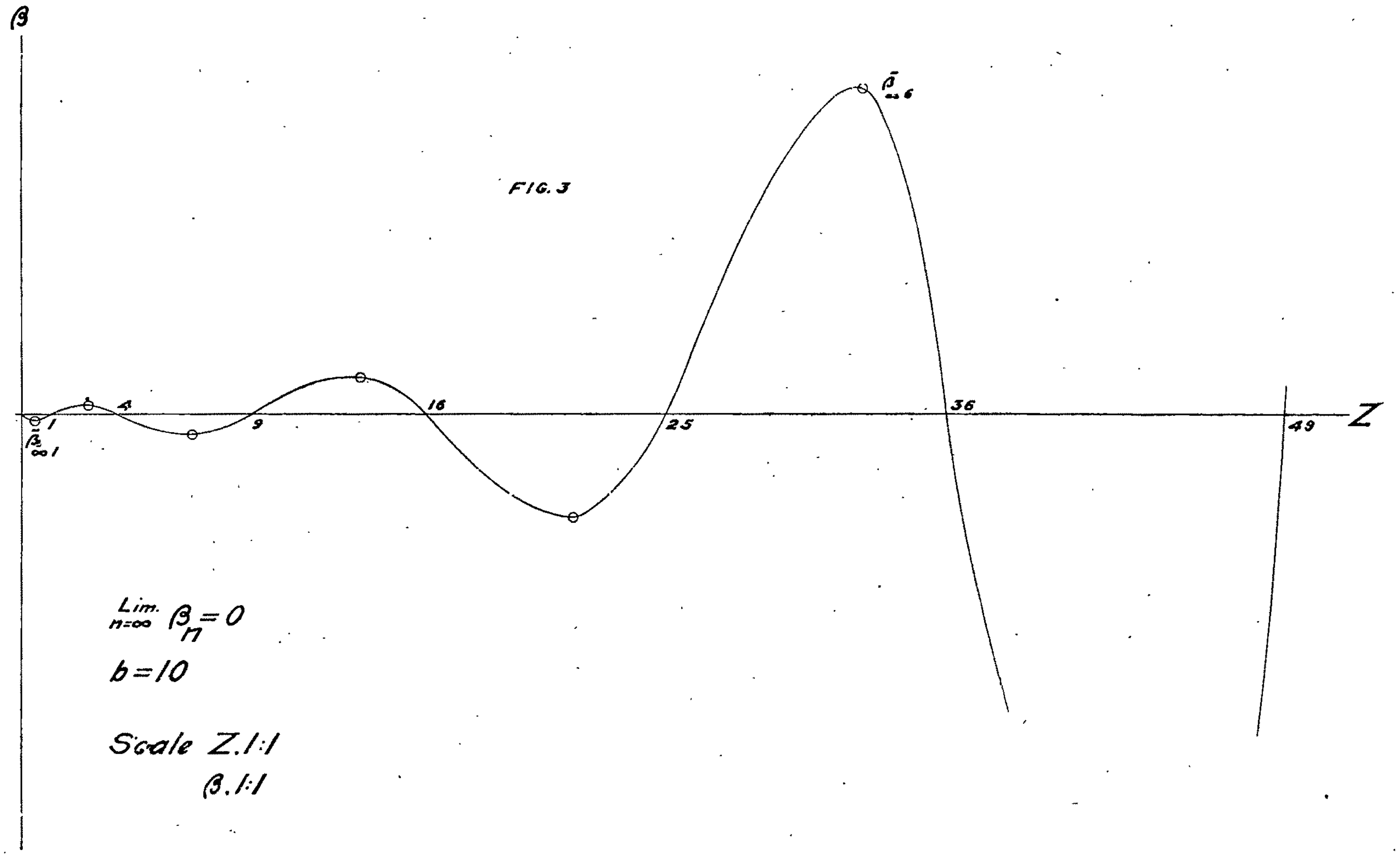


FIG. 3

β
Z
1
4
9
16
25
36
49
$\bar{\beta}_{\infty 1}$
$\bar{\beta}_{\infty 6}$
$\lim_{n=\infty} \beta_n = 0$
$b = 10$
Scale Z. 1:1
β. 1:1

By similar reasoning, the first maximum $\bar{\beta}_{\infty 2}$ is $+0.33$, approximately. By further computation and reasoning from the recursion formula, it is found that the succeeding minima and maxima of β_n converge to limits for $n=\infty$, each larger than the preceding numerically, and all differing from zero.

The general form of the curve $\beta_\infty = 0$ is shown by Fig. 3, the nullpoints being determined by $z_{\infty 1}, z_{\infty 2}, \ldots, z_{\infty 7}$, as subsequently computed and given in Table II.

II. *Argument* $b = 100$.

For argument $b = 100$, $\beta_\infty = 0$ has nullpoints still nearer 0, 1, 4, 9, ..., $(n-1)^2$, as shown in Table III. The locus of $\beta_\infty = 0$, for $b = 100$, has the same general form as Fig. 3, the first minimum and the first maximum being nearly the same as for $b = 10$. In no case is a maximum or minimum zero.

III. *Argument* $b = 0.1$.

The maxima and minima values of β_∞ for $b = 0.1$ present greater difficulties, due to the fact that $(0.1)^2$ occurs in all values of β_n and to the fact that the roots of $\beta_n = 0$ do not fall in the regular intervals -2, 1, 4, 9, ..., $(n-1)^2$, n^2 until

$$b(n-2) > 1. \tag{18}$$

See Table I and Heine's *Kugelfunktionen*, I, 407.

To determine the laws governing these maxima and minima values, we compute $\bar{z}$ and $\bar{\beta}$ found in Table IV, using the following equations:

$$\beta_2 = +\frac{1}{2}\left(z^2 - z - \frac{2}{b^2}\right) = +\frac{1}{2}(z^2 - z - 200), \tag{19}$$

$$\beta_3 = -\frac{1}{8}(z^3 - 5z^2 - 296z + 800), \tag{20}$$

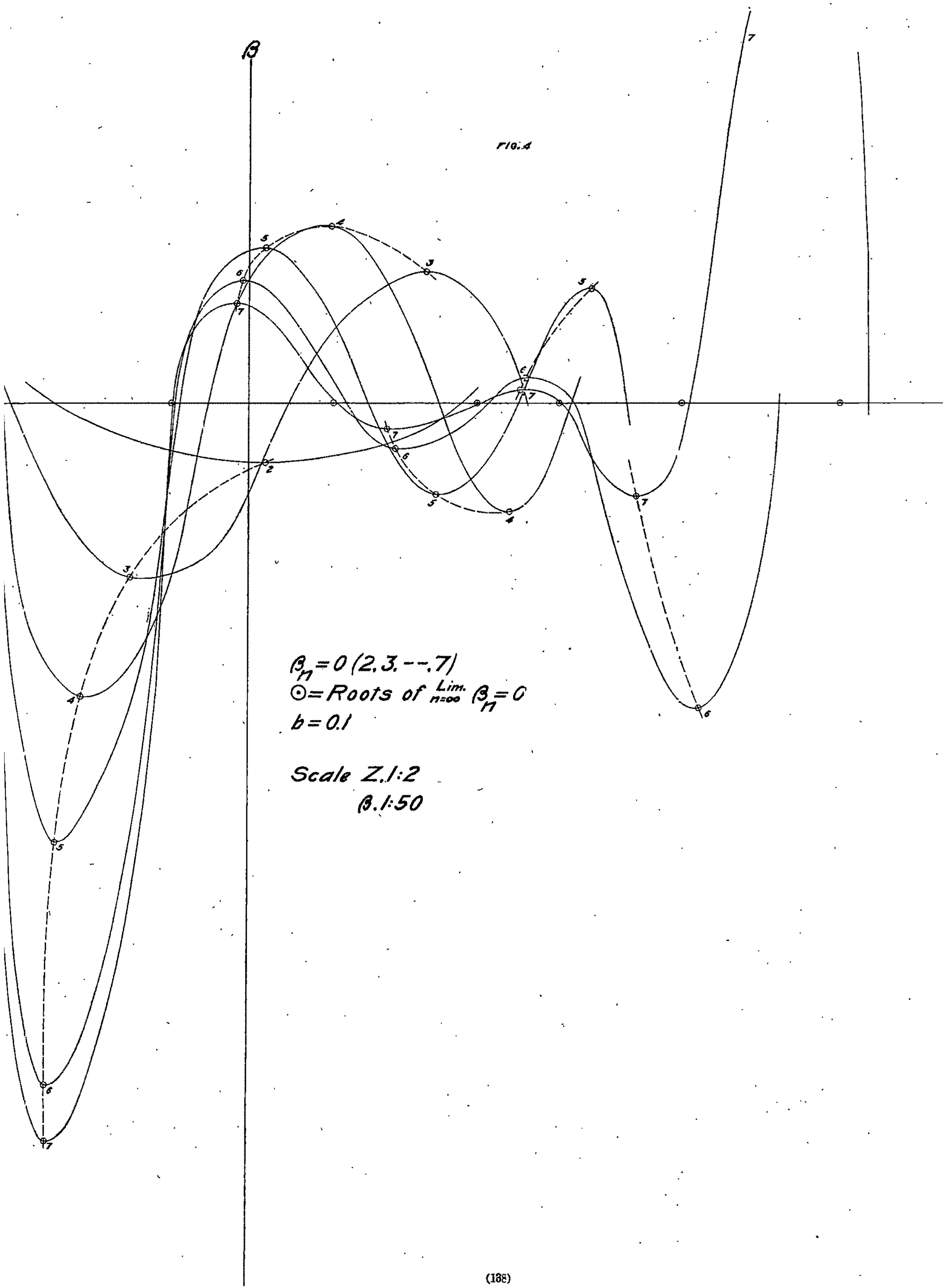
$$\beta_4 = +\frac{1}{72}(z^4 - 14z^3 - 351z^2 + 3564z + 12800), \tag{21}$$

$$\beta_5 = -\frac{1}{1152}(z^5 - 30z^4 - 227z^3 + 9680z^2 - 14624z - 284800), \tag{22}$$

$$\beta_6 = +\frac{1}{28800}(z^6 - 55z^5 + 423z^4 + 16755z^3 - 221524z^2 - 275600z + 5840000), \tag{23}$$

$$\beta_7 = -\frac{1}{1036800}(z^7 - 91z^6 + 2303z^5 + 4527z^4 - 802004z^3 + 6731264z^2 + 17224000z - 181760000), \tag{24}$$

FIG. 4


β
$\beta_n = 0\ (2, 3, --, 7)$
⊙ = Roots of $\lim_{n=\infty} \beta_n = 0$
$b = 0.1$
Scale Z. 1:2
β. 1:50
2
3
4
5
6
7

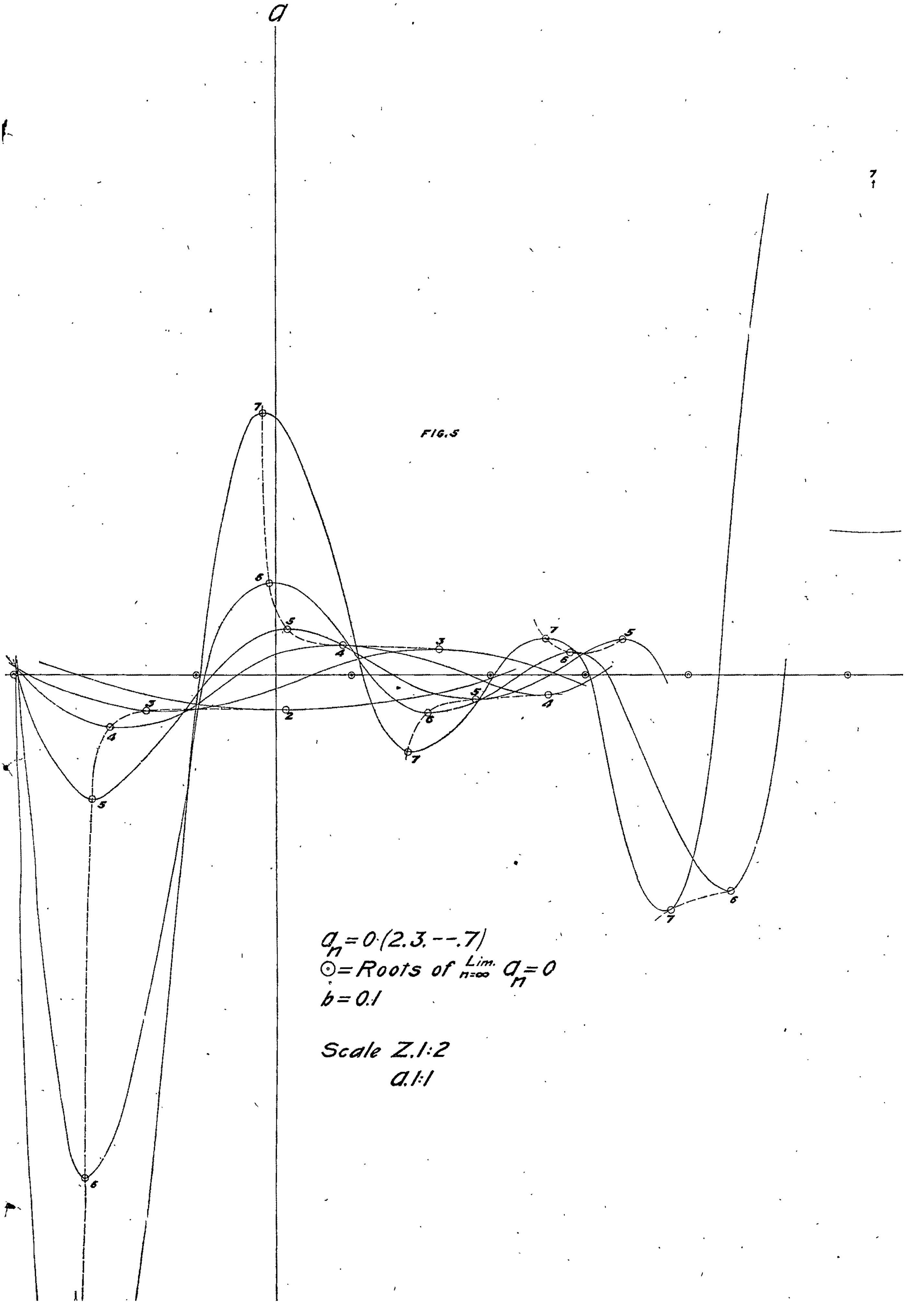


FIG. 5

$a_n = 0\cdot(2.3.--.7)$

⊙ = Roots of $\lim_{n=\infty} a_n = 0$

$b = 0.1$

Scale Z. 1:2

a. 1:1

$$(25)\quad \beta_2' = +1\left(z - \tfrac{1}{2}\right) = 0,$$

$$(26)\quad \beta_3' = -\frac{3}{8}\left(z^2 - 3\tfrac{1}{3}z - 93\tfrac{2}{3}\right) = 0,$$

$$(27)\quad \beta_4' = +\frac{4}{72}\left(z^3 - 10\tfrac{1}{2}z^2 - 175\tfrac{1}{2}z + 891\right) = 0,$$

$$(28)\quad \beta_5' = -\frac{5}{1152}\left(z^4 - 24z^3 - 136\tfrac{1}{5}z^2 + 3872z - 2924\tfrac{4}{5}\right) = 0,$$

$$(29)\quad \beta_6' = +\frac{6}{28800}\left(z^5 - 45\tfrac{5}{6}z^4 + 282z^3 + 8377\tfrac{1}{2}z^2 - 73841\tfrac{1}{3}z - 45933\tfrac{1}{3}\right) = 0,$$

$$(30)\quad \beta_7' = -\frac{7}{1036800}\left(z^6 - 78z^5 + 1645z^4 + 2301\tfrac{1}{7}z^3 - 343716z^2 + 1923218\tfrac{2}{7}z + 2460571\tfrac{3}{7}\right) = 0.$$

In Fig. 4, we locate the nullpoints of $\beta_\infty = 0$ from the values of $a_{\infty 1}$ to $a_{\infty 7}$ in Table I, and draw the curves representing $\beta_2 = 0$ to $\beta_7 = 0$.

In studying these curves the following tendencies should be considered (see Table I and Fig. 4):

a) The nullpoints of these curves always approach the nullpoints of $\beta_\infty = 0$ as n increases.

b) The first $\bar{z}_{n1}$ and the corresponding minima values of β always increase numerically with n. $\lim\limits_{n=\infty} \bar{\beta}_{n1}$ must be examined.

c) The first maximum increases from $\beta_3 = 0$ to $\beta_4 = 0$ and afterwards decreases, apparently to a small positive limit for $\beta_\infty = 0$.

d) The other minima and maxima between two successive roots of $\beta_n = 0$ always decrease numerically as n increases, apparently toward a very small limit for $\beta_\infty = 0$.

e) In each curve, the maxima and minima for the first half of the arches retrograde and for the last half advance beyond the middle of the interval.

To discuss these tendencies and eventually to discover properties of $\beta_\infty = 0$, return to the formula

$$a_n = b\left((n-1)^2 - z\right)a_{n-1} - a_{n-2}.$$

Substituting $a_n = b^n (n-1)!^2 \beta_n$ gives

$$(31)\qquad \beta_n = \left(1 - \frac{z}{(n-1)^2}\right)\beta_{n-1} - \frac{1}{b^2(n-1)^2(n-2)^2}\beta_{n-2}.$$

First Minimum.

To explain tendency b) and to determine the limit of the first minimum, use equation (31) and Table IV. The first minimum evidently lies between $z_{\infty 1} = -16.9015$ and $z_{\infty 2} = -5.0524$, the first two roots of $a_\infty = 0$ and of $\beta_\infty = 0$.

Beginning with $\beta_5 = 0$,

$$\bar{\beta}_{51} = \left(1 + \frac{|\bar{z}_{51}|}{4^2}\right)\beta_{41} - \frac{1}{0.01 \times 4^2 \times 3^2}\beta_{31}, \text{ for } \bar{z}_{51} = -12.6, \tag{32}$$

$$\bar{\beta}_{61} = \left(1 + \frac{|\bar{z}_{61}|}{5^2}\right)\beta_{51} - \frac{1}{0.01 \times 5^2 \times 4^2}\beta_{41}, \text{ for } \bar{z}_{61} = -13.0, \tag{33}$$

$$\bar{\beta}_{71} = \left(1 + \frac{|\bar{z}_{71}|}{6^2}\right)\beta_{61} - \frac{1}{0.01 \times 6^2 \times 5^2}\beta_{51}, \text{ for } \bar{z}_{71} = -13.3, \tag{34}$$

$$\bar{\beta}_{n1} = \left(1 + \frac{|\bar{z}_{n1}|}{(n-1)^2}\right)\beta_{n-1,1} - \frac{1}{0.01\,(n-1)^2(n-2)^2}\beta_{n-2}, \text{ for } \bar{z}_{n1}. \tag{35}$$

Since $\bar{\beta}_{51}$, $\bar{\beta}_{61}$, $\bar{\beta}_{71}$ are known by computation to be negative and $|\bar{\beta}_{51}| < |\bar{\beta}_{61}| < |\bar{\beta}_{71}|$, it is evident from (35) that $|\bar{\beta}_{81}| < |\bar{\beta}_{91}| < \cdots\cdot < |\bar{\beta}_{\infty 1}|$, and that they are all negative, since the first term in the second member of (35) has the greater multiplier from $\bar{\beta}_{51}$ to $\bar{\beta}_{\infty 1}$, the last multiplier being a proper fraction and decreasing rapidly while the first multiplier $1 + \frac{|\bar{z}_{n1}|}{(n-1)^2}$ remains greater than unity.

The relation $|\bar{\beta}_{71}| < |\bar{\beta}_{81}| \cdots\cdot < |\bar{\beta}_{\infty 1}|$ is preserved when $\bar{z}_{\infty 1}$ is substituted for $\bar{z}_{71}$, $\bar{z}_{81}, \ldots\,$, and the inequalities are still greater, since less ordinates are substituted for maximum ordinates.

(36) However,

$$|\bar{\beta}_n| < \left|\left(1 + \frac{|\bar{z}_n|}{(n-1)^2}\right)\beta_{n-1}\right| \text{ and } |\beta_{n-1}| < \left|\left(1 + \frac{|\bar{z}_{n-1}|}{(n-2)^2}\right)\beta_{n-2}\right|,$$

since $\left|\frac{1}{0.01\,(n-1)^2\,(n-2)^2}\beta_{n-1}\right|$ has some small positive value while n is finite.

(37) Hence

$$|\bar{\beta}_{n1}| < \left(1 + \frac{|\bar{z}_{n1}|}{(n-1)^2}\right) \times \left(1 + \frac{|\bar{z}_{n1}|}{(n-2)^2}\right) \times \cdots\cdot \times \left(1 + \frac{|\bar{z}_{n1}|}{7^2}\right) \times |\beta_{71}|,$$

for $\bar{z}_{\infty 1}$. For $n = \infty$, $z_{n1} = -13.5$ approximately, the last figure being determined by the recursion formula. Computing $\beta_{71} = 1224.4$ for $\bar{z}_{\infty 1} = -13.5$ by Horner's method,

$$(38)\qquad \lim_{n=\infty} |\overline{\beta}_{n1}| < \left(1 + \frac{13.5}{(n-1)^2}\right) \times \left(1 + \frac{13.5}{(n-2)^2}\right) \times \ldots \times \left(1 + \frac{13.5}{7^2}\right) \times 1224.4.$$

For an infinite product,

$$(39)\qquad \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2 \pi^2}\right) = \frac{e^z - e^{-z}}{2z}. \qquad \frac{z^2}{\pi^2} = 13.5,\ z = \pi\sqrt{13.5}.$$

$$(40)\qquad |\overline{\beta}_{\infty 1}| < 1224.4 \,.\, \frac{e^{\pi\sqrt{13.5}} - e^{-\pi\sqrt{-13.5}}}{2\pi\sqrt{13.5}} \div \prod_{k=1}^{6} \left(1 + \frac{13.5}{k^2}\right).$$

$$(41)\qquad \overline{\beta}_{\infty 1} = -8826.7.$$

The first minimum is therefore a finite number.

Second Minimum.

The second minimum first appears in $\beta_4 = 0$, and by Table IV,

$$(42)\qquad |\overline{\beta}_{53}| < |\overline{\beta}_{43}|,$$

and both are negative.

By equation (31) and Table IV,

$$(43)\qquad \overline{\beta}_{63} = \left(1 - \frac{9.5}{25}\right)\beta_{53} - \frac{1}{0.01 \,.\, 25 \,.\, 16}\beta_{43}, \text{ for } \bar{z}_{33} = +9.5;$$

$$(44)\qquad \overline{\beta}_{73} = \left(1 - \frac{9.5}{36}\right)\beta_{63} - \frac{1}{0.01 \,.\, 36 \,.\, 25}\beta_{53}, \text{ for } \bar{z}_{73} = +9.5;$$

$$(45)\qquad \overline{\beta}_{83} = \left(1 - \frac{9.4}{49}\right)\beta_{73} - \frac{1}{0.01 \,.\, 49 \,.\, 36}\beta_{63}, \text{ for } \bar{z}_{33} = +9.4.$$

By computing the coefficients of (43), (44), (45), it is found that β_{n3} continues negative and of decreasing numerical value. Since the second term of the second member is decreasing rapidly, on account of the factors $(n-1)^2 (n-2)^2$ in the denominator, toward the limit zero, and $1 - \frac{\bar{z}}{(n-1)^2}$ more slowly increases toward the limit unity, it is seen that $\overline{\beta}_{83}$ equals β_{73} multiplied by a positive proper fraction, plus a much smaller number. Thus $\overline{\beta}_{83}, \overline{\beta}_{93}, \ldots, \overline{\beta}_{\infty 3}$ remain negative and $|\overline{\beta}_{83}| > |\overline{\beta}_{93}| \ldots > |\overline{\beta}_{\infty 3}|$.

$\lim_{n=\infty} \overline{\beta}_{n,3} = \overline{\beta}_{n-1,3}$, and $\overline{\beta}_{\infty}$ is a small negative number. By repeated use of (31),

$$(46)\qquad \overline{\beta}_{\infty 3} = -27. \qquad \text{See Fig. 4.}$$

Third and Subsequent Minima.

The third and subsequent minima can be treated in the same manner. We observe that the first minimum in any set or range may be large as compared with the minima in the preceding sets. For example, $\overline{\beta}_{65}$, for $\bar{z}_{65} = +29.8$, is -497.6. This irregularity is due to the fact that the first minimum in a set lies between the last two roots of an equation of an even degree; e. g., $\beta_6 = 0$. For $b < 1$, as in the case under consideration, the last root does not, in general, lie between $(n-1)^2$ and n^2 until $b(n-1) > 1$.*

In such a case $\bar{z}$ may be greater than $(n-1)^2$, as in $\bar{z}_{65}$. Hence

(47) $\overline{\beta}_{65} = \left(1 - \frac{\bar{z}_{65}}{(n-1)^2}\right)\beta_{55} - \frac{1}{0.01\,(n-1)^2\,(n-2)^2}\,\beta_{45}$ has irregularities. By computation,

$$\overline{\beta}_{65} = \left(1 - \frac{29.8}{25}\right)(-1673) - \tfrac{1}{4}(+3275.3), \text{ for } \bar{z}_{65} = +29.8. \tag{48}$$

$$\overline{\beta}_{65} = (1 - 1.192)(-1673) - \tfrac{1}{4}(+3275.3) = -497.6 \text{ approximately.} \tag{49}$$

The first parenthesis changes sign, since $\bar{z}_{65} > (n-1)^2$, and $\overline{\beta}_{65}$ therefore becomes a large positive number. However, the coefficient of the next term is comparatively large, so that the last term controls the sign of $\overline{\beta}_{65}$.

However,

$$\overline{\beta}_{75} = \left(1 - \frac{25.7}{36}\right)\beta_{65} - \frac{1}{9}\,\beta_{55} = -152.1, \text{ for } \bar{z}_{75} = 25.7. \tag{50}$$

Here and subsequently in $\overline{\beta}_{85}, \ldots, \overline{\beta}_{\infty 5}$ the parenthesis is positive, slowly approaching the limit unity, and both terms are negative. As the last coefficient decreases rapidly toward the limit zero, the values of $\overline{\beta}_{85}, \ldots, \overline{\beta}_{\infty 5}$ decrease numerically with increasing n and are always negative. Hence, as in the second minima,

$$\lim_{n=\infty} \overline{\beta}_{n5} = \overline{\beta}_{n-1,5} \tag{51}$$

and $\overline{\beta}_{\infty 5}$ is a very small negative number, and not zero.

The same argument applies to subsequent sets of minima values.

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* Heine's *Kugelfunktionen*, I, 407.

$$\text{(59)}\qquad \text{For } z = 5.58135, \qquad \begin{aligned} a_6 &= +\,0.228476975 \\ a_7 &= +\,0.054793576 \\ a_8 &= +\,0.009429440 \\ a_9 &= +\,0.000197600 \\ a_{10} &= -\,0.007939170 \end{aligned}$$

Here a_9 is positive but a_{10} is negative, and $z = 5.58135$ locates a point in the z-axis between the nullpoints of $a_9 = 0$ and $a_{10} = 0$.

Computing $a_1, a_2, \ldots, a_7$ by Horner's method,

$$\text{(60)}\qquad \text{For } z = 5.58134, \qquad \begin{aligned} a_1 &= -\,0.279067000 \\ a_2 &= -\,0.872149970 \\ a_3 &= +\,0.416988556 \\ a_4 &= +\,1.014702419 \\ a_5 &= +\,0.640200395 \\ a_6 &= +\,0.228482548 \\ a_7 &= +\,0.054808073 \\ a_8 &= +\,0.009687228 \\ a_9 &= +\,0.001783414 \\ a_{10} &= +\,0.003763039 \end{aligned}$$

It is evident that 5.58134 is the value of $z_{\infty 3}$, correct to five decimal places; since $a_{10} < a_{11} < a_{12} \ldots\ldots < a_{\infty}$ by (56), and all the curves $a_{10} = 0$ to $a_{\infty} = 0$ are still above the z-axis for $z = 5.58134$, but below the z-axis for $z = 5.58135$.

Substituting (60) in (55),

$$\text{(61)}\qquad \begin{aligned} E(\phi) = 0.5 &- 0.2790 \cos 2\phi - 0.8721 \cos 4\phi + 0.4169 \cos 6\phi \\ &+ 1.0147 \cos 8\phi + 0.6402 \cos 10\phi + 0.2284 \cos 12\phi \\ &+ 0.0547 \cos 14\phi + 0.0096 \cos 16\phi + 0.0017 \cos 18\phi \\ &+ 0.0037 \cos 20\phi + \ldots\ldots \end{aligned}$$

The accuracy of these coefficients is proved by the recursion formula (56).

Convergence of the Series.

This series is computed for a root $z_{\infty 3}$, accurate to the fifth decimal place. By a careful consideration of the last remainders in Horner's process used in computing $a_1, a_2, \ldots, a_6$ it is evident that the first three decimal figures in these coefficients will never change, if an infinite number of figures of $z_{\infty 3}$ are computed and substituted in $E(\phi)$, since an increase of a whole unit in the sixth figure of z does not cause a change in the first three decimal places of a_1, a_2, $\ldots$, a_6.

Calling the finite sum of the first four terms F, we write,

(62) $$|E(\phi)| < |F \pm (1.0147 + 0.6402 + 0.2284 + 0.0548 + 0.0096 + \ldots)|,$$

substituting maximum values for $\cos 8\phi$, $\cos 10\phi$, $\ldots$, and using the +sign when F and the following series have like signs, and the —sign when they have opposite signs.

We must now define a root of $a_\infty = 0$ more carefully.

(63) *Definition of a Root of* $a_\infty = 0$.

From a certain a_n onward indefinitely, for an exact root of $a_\infty = 0$, a_{n+1} is less than a_n and has the same sign to $n = \infty$.

Otherwise one of the following relations must exist:

(64) 1) $a_{n+1} > a_n$, with the same sign.
(65) 2) a_{n+1} and a_n differ in sign.
(66) 3) $a_{n+1} < a_n$ for one or more terms and then $a_{n+1} > a_n$.

Supposition (64) can not be true for $z_{\infty 1}$ a root of $a_\infty = 0$, since a_{n+2}, a_{n+3}, $\ldots$, a_∞ would form an increasing series, as is shown by the formula

(67) $$a_{n+2} = b\left((n+1)^2 - z\right) a_{n+1} - a_n.$$

When $b\left((n+1)^2 - z\right)$ becomes greater than 2 with increasing n and $a_{n+1} > a_n$, a_{n+2} must be greater than a_{n+1}. Hence $\lim\limits_{n=\infty} a_n$ would not be zero.

The second supposition (65) is false for a root of $a_\infty = 0$, because (67) shows that, when a_{n+1} and a_n have opposite signs and the coefficient of a_{n+1} becomes and remains positive with increasing n, a_{n+2} must, under condition (65), have a larger absolute value than a_{n+1}. Hence $\lim\limits_{n=\infty} a_n$ is not zero.

The third supposition (66) reduces to (64) or to (65) and therefcre can not be true.

Hence (63) defines a root of $a_\infty = 0$.

This definition gives the following rule:

(68) *Rule for Computing Roots of* $a_\infty = 0$.

Find the successive figures of positive roots of $a_\infty = 0$ as great as possible, so that a_6, a_7, a_8, $\ldots$, a_∞ shall have the same signs.

For negative roots, each succeeding figure of $z_{\infty 1}$ is one less than the least figure that gives a permanence in the signs of a_6, a_7, a_8, $\dots$, a_∞.

For a root of $a_\infty = 0$, as above defined, (67) gives

$$a_{n+2} = b\left((n+1)^2 - z\right) a_{n+1} - a_n < a_{n+1},$$

$$(69) \qquad a_{n+1} < \frac{a_n}{b\left((n+1)^2 - z\right) - 1}.$$

From a certain n onward to $n = \infty$,

$$(70) \qquad a_{n+1} < \tfrac{1}{2} a_n.$$

Therefore the last series in (62) may be written

$$(71) \qquad 1.0147 + 0.6402 + 0.2284 + \dots < 1.0147 + 0.6402\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots\right),$$

which is clearly a converging series.

Hence from (62) and (71), when $z_{\infty 3}$ is an exact root of $a_\infty = 0$,

$$(72) \qquad E(\phi) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi \text{ is finite and the series is convergent.}$$

II. *Argument* $b = 10$.

$$(73) \qquad E(\phi) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi.$$

The computation of the values of a_n with sufficient accuracy to show that a_n converges toward the limit zero and that the series representing $E(\phi)$ is convergent when n increases without limit, involves difficulties due to the rapidity with which the curves $a_n = 0$ approach perpendicularity to the z-axis at the nullpoints.

For example, we compute the third root of $a_7 = 0$ by Horner's method, carrying the remainders to eighteen decimal places, and find the values of a_6 and a_7 for this root as we obtain the successive approximations. These computations are continued until the corresponding values of a_{63} and a_{73} give a negative value for a_{83} by the recursion formula.

We find $z_{73} = 4.00133 +$, and observe from the recursion formula that

$$a_8 = 10\,(49 - 4.00133)\, a_7 - a_6 = 450\, a_7 - a_6, \text{ approximately.}$$

Evidently a_8 will not become negative until $450\,a_7$ is less than a_6. When these values of a_6 and a_7 are found, $z_{\infty\,3}$ can be found as follows:

	z_{73}	a_{73}	a_{63}
(74)	4	3232143580.000000	10100579.000000
	0	"	"
	0	"	"
	1	814029735.729124	2543960.282771
	3	88386514.036511	276223.188814
	3	15816887.536327	49430.563659
	6	1302846.520815	4076.625965
	5	93341.361791	291.709237
	3	20771.043732	64.914147
	8	1418.958754	4.435519
	5	208.453440	0.655601
	8	15.932590	0.050815
	6	1.418526	0.005456
	5	0.209021	0.001676
	8	0.015500	0.001071
	6	0.000956	0.001026
	3	0.000231	0.001023
	9	0 000013	0.001023
	5	0.000001	0.001023

Substituting the last results in (67), using z_{73} to eighteen decimal places,

$$(75)\qquad \begin{aligned} a_8 &= 10\,(49 - 4.001336538586586395)\,.\,0.0000012 - 0.0010230 \\ &= -\,0.000490. \end{aligned}$$

Since a_8 is negative, z is too large by (63).

Taking z to seventeen decimal places,

$$(76)\qquad \begin{aligned} a_8 &= 10\,(49 - 4.00133653858658639)\,.\,0.0000133 - 0.0010231 \\ &= +\,0.0049617. \end{aligned}$$

Since a_8 is here positive and in the former case negative, $z_{\infty\,3}$ lies between the two values taken.

$$z_{\infty\,3} = 4.00133653858658639 + .$$

Subsequent figures may be found as in (57) to (60).

$$+ \; 1.142740 \cos 20\phi + 0.166280 \cos 22\phi + \ldots\ldots$$

It will be noticed in these equations, that, in general, the smaller the roots are algebraically the better are the coefficients obtained for an approximate solution in the form of a finite number of terms of the infinite series. From Fig. 5, we should expect the best approximations for E_5 and E_4; and this is doubtless in general true, since the maxima and minima are here less numerically and the slopes of the curves at the nullpoints are not so great, but the possible difference in the fifth and subsequent decimal places of $z_{\infty\,1}, \ldots, z_{\infty\,7}$ make it impossible to determine this fact definitely without further calculations.

To obtain, for $b = 10$, as good approximate results as the above, the values of $z_{\infty\,1}, \ldots, z_{\infty\,7}$ must be computed to eighteen or more decimal places and, for $b = 100$, to many more places.

Hence,

(77) $E(\phi) = 0.5 + \sum_{n=1}^{5} a_n \cos 2n\phi + 0.0010231 \cos 12\phi + 0.0000013 \cos 14\phi + \dots\dots$

The trial divisor in Horner's process contained twenty figures when the contract method was used in determining the last fifteen figures of z. To secure values for a_1 to a_8 that will satisfy the recursion formula, twenty-one decimal places should be used in all remainders and the contract method should be employed later in the work.

It will be noted that even with seventeen decimal places of $z_{\infty 3}$ accurately computed, a_{83} is greater than a_{73}. It is evidently practically impossible to find a value for $z_{\infty 3}$ so near the exact root that a_{73}, a_{83}, $\dots$, $a_{\infty 3}$ shall form a rapidly decreasing series. Heine* remarks that when such a value is found,

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As a general conclusion, it is evident that small values of b and z are the best values.

General Proof of Convergence.

Granting that $a_\infty = f(b, z) = 0$ is an equation of an infinite degree in the general form of equations (5) and that the coefficients of the series in (2) satisfy the recursion formula (3) for exact roots of $a_\infty = 0$, we have shown that these roots can be computed to any desired degree of accuracy by rule (68). We must now show that the series in the solution $E(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi$ is convergent for values of a_n computed for $z_{\infty i}$, any exact root of $a_\infty = 0$.

Beginning with a certain a_m, definition (63) gives

$$a_{m+1} < \frac{a_m}{b\left((m+1)^2 - z\right) - 1}, \tag{69}$$

$$a_{m+1} < \tfrac{1}{2} a_m. \tag{70}$$

(82) $$\begin{aligned} E(\phi) &= \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi \\ &= (\tfrac{1}{2} a_0 + a_1 \cos 2\phi + \dots\dots + a_{m-1} \cos 2(m-1)\phi) \\ &\quad + (a_m \cos 2m\phi + \dots\dots + a_\infty \cos \infty\phi). \end{aligned}$$

If a_m is the first term of the decreasing series that characterizes a root of $a_\infty = 0$ in definition (63) and satisfies relation (70),

$$|E(\phi)| < |(\tfrac{1}{2} a_0 + a_1 \cos 2\phi + \dots\dots + a_{m-1} \cos 2(m-1)\phi) \pm (a_m + a_{m+1} + \dots\dots + a_\infty)|,$$

substituting maximum values for $\cos 2m\phi, \dots\dots,$ and using the plus sign when the two parentheses are alike in sign, and the minus sign when they are of opposite sign.

$$|a_m + a_{m+1} + \dots\dots + a_\infty| < |a_m(1 + \tfrac{1}{2} + \tfrac{1}{4} + \dots\dots)| = |2a_m|. \tag{70}$$

The quantities $a_1, a_2, \dots\dots, a_m$ are finite, since a_n is a rational, integral function of z with finite coefficients; hence $a_1, a_2, \dots\dots, a_m$ must be finite for finite values of z.

Hence,

$$E(\phi)| < |(\text{Finite Sum} \pm 2a_m)|, \text{ and}$$

$E(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\phi$ is a convergent series and a solution of

$$\frac{d^2E}{d\phi^2} + \left(\frac{8}{b} \cos 2\phi + 4z\right) E = 0.$$

TABLE I.

$b = 0.1$

Z_{11}	+ 0.0000	A_{11}	+0.00000
Z_{21}	−13.6509	A_{21}	−0.00000
Z_{22}	+14.6509	A_{22}	−0.00000
Z_{31}	−16.2479	A_{31}	−0.00002
Z_{32}	+ 2.6470	A_{32}	−0.00000
Z_{33}	+18.6009	A_{33}	+0.00002
Z_{41}	−16.8064	A_{41}	−0.00000
Z_{42}	− 2.8853	A_{42}	+0.00002
Z_{43}	+12.3932	A_{43}	+0.00000
Z_{44}	+21.2985	A_{44}	−0.00001
Z_{51}	−16.8934	A_{51}	−0.00015
Z_{52}	− 4.6582	A_{52}	+0.00003
Z_{53}	+ 7.7970	A_{53}	+0.00001
Z_{54}	+18.0760	A_{54}	−0.00002
Z_{55}	+25.6784	A_{55}	+0.00034
Z_{61}	−16.9011	A_{61}	−0.00013
Z_{62}	− 5.0133	A_{62}	+0.00008
Z_{63}	+ 5.9979	A_{63}	+0.00004
Z_{64}	+15.7525	A_{64}	−0.00002
Z_{65}	+21.9884	A_{65}	+0.00003
Z_{66}	+33.1754	A_{66}	−0.00074
Z_{71}	−16.9015	A_{71}	−0.00109
Z_{72}	− 5.0500	A_{72}	+0.00008
Z_{73}	+ 5.6193	A_{73}	+0.00000
Z_{74}	+14.7525	A_{74}	−0.00004
Z_{75}	+20.7688	A_{75}	+0.00006
Z_{76}	+28.6580	A_{76}	−0.00021
Z_{77}	+43.1526	A_{77}	+0.00348
$Z_{\infty 1}$	−16.9015		
$Z_{\infty 2}$	− 5.0524		
$Z_{\infty 3}$	+ 5.5813		
$Z_{\infty 4}$	+14.5616		
$Z_{\infty 5}$	+20.4705		
$Z_{\infty 6}$	+27.1931		
$Z_{\infty 7}$	+37.3476		

TABLE II.

$b = 10$

Z_{11}	+ 0.0000	A_{11}	+ 0.00000
			$\times 10^{1}$
Z_{21}	− 0.0196	A_{21}	− 0.00002
Z_{22}	+ 1.0196	A_{22}	− 0.00002
			$\times 10^{2}$
Z_{31}	− 0.0196	A_{31}	− 0.00013
Z_{32}	+ 1.0163	A_{32}	− 0.00006
Z_{33}	+ 4.0033	A_{33}	+ 0.00071
			$\times 10^{3}$
Z_{41}	− 0.0196	A_{41}	− 0.00112
Z_{42}	+ 1.0163	A_{42}	− 0.00032
Z_{43}	+ 4.0013	A_{43}	+ 0.00110
Z_{44}	+ 9.0019	A_{44}	− 0.01295
			$\times 10^{4}$
Z_{51}	− 0.0196	A_{51}	− 0.01875
Z_{52}	+ 1.0163	A_{52}	− 0.00476
Z_{53}	+ 4.0013	A_{53}	+ 0.01315
Z_{54}	+ 9.0005	A_{54}	− 0.09021
Z_{55}	+16.0014	A_{55}	+ 0.28200
			$\times 10^{5}$
Z_{61}	− 0.0196	A_{61}	− 0.52699
Z_{62}	+ 1.0163	A_{62}	− 0.08015
Z_{63}	+ 4.0013	A_{63}	+ 0.27626
Z_{64}	+ 9.0005	A_{64}	− 1.44069
Z_{65}	+16.0003	A_{65}	+ 1.59149
Z_{66}	+25.0011	A_{66}	−10.02866
			$\times 10^{6}$
Z_{71}	− 0.0196	A_{71}	− 16.90346
Z_{72}	+ 1.0163	A_{72}	− 3.86381
Z_{73}	+ 4.0013	A_{73}	+ 3.33886
Z_{74}	+ 9.0005	A_{74}	− 7.26793
Z_{75}	+16.0003	A_{75}	+ 31.69049
Z_{76}	+25.0002	A_{76}	−196.55479
Z_{77}	+36.0009	A_{77}	+1084.83159
			$\times 10^{7}$
$Z_{\infty 1}$	− 0.0196		
$Z_{\infty 2}$	+ 1.0163		
$Z_{\infty 3}$	+ 4.0013		
$Z_{\infty 4}$	+ 9.0005		
$Z_{\infty 5}$	+16.0003		
$Z_{\infty 6}$	+25.0002		
$Z_{\infty 7}$	+36.0001		

TABLE III.

$b = 100$

Z_{11}	+ 0.0000	A_{11}	+ 0.00000
			$\times 10^{2}$
Z_{21}	− 0.0001	A_{21}	− 0.00004
Z_{22}	+ 1.0001	A_{22}	− 0.00004
			$\times 10^{4}$
Z_{31}	− 0.0001	A_{31}	− 0.00019
Z_{32}	+ 1.0001	A_{32}	− 0.00009
Z_{33}	+ 4.0000	A_{33}	+ 0.00020
			$\times 10^{6}$
Z_{41}	− 0.0001	A_{41}	− 0.00179
Z_{42}	+ 1.0001	A_{42}	− 0.00079
Z_{43}	+ 4.0000	A_{43}	+ 0.00040
Z_{44}	+ 9.0000	A_{44}	− 0.00359
			$\times 10^{8}$
Z_{51}	− 0.0001	A_{51}	− 0.02879
Z_{52}	+ 1.0001	A_{52}	− 0.01199
Z_{53}	+ 4.0000	A_{53}	+ 0.00480
Z_{54}	+ 9.0000	A_{54}	− 0.00719
Z_{55}	+16.0000	A_{55}	+ 0.14400
			$\times 10^{10}$
Z_{61}	− 0.0001	A_{61}	− 0.71997
Z_{62}	+ 1.0001	A_{62}	− 0.28791
Z_{63}	+ 4.0000	A_{63}	+ 0.10080
Z_{64}	+ 9.0000	A_{64}	− 0.11520
Z_{65}	+16.0000	A_{65}	+ 0.28798
Z_{66}	+25.0000	A_{66}	− 8.83001
			$\times 10^{12}$
Z_{71}	− 0.0001	A_{71}	−25.91922
Z_{72}	+ 1.0001	A_{72}	−10.07714
Z_{73}	+ 4.0000	A_{73}	+ 3.22567
Z_{74}	+ 9.0000	A_{74}	− 3.11034
Z_{75}	+16.0000	A_{75}	+ 5.76028
Z_{76}	+25.0000	A_{76}	−20.16004
Z_{77}	+36.0000	A_{77}	+1088.63940
			$\times 10^{14}$
$Z_{\infty 1}$	− 0.0001		
$Z_{\infty 2}$	+ 1.0000		
$Z_{\infty 3}$	+ 4.0000		
$Z_{\infty 4}$	+ 9.0000		
$Z_{\infty 5}$	+16.0000		
$Z_{\infty 6}$	+25.0000		
$Z_{\infty 7}$	+36.0000		

TABLE IV.

$b = 0.1$		$+\bar{\beta} = \text{Max.}$ $-\bar{\beta} = \text{Min.}$		
$a_n = 0, \beta_n = 0$		$\bar{z}$	$\bar{\beta}$	$\bar{a}$
Z_{21}	-13.6509	$+0.5$	-100.1	-1.00100
Z_{22}	$+14.6509$			
Z_{31}	-16.2479	-8.4	-292.6	-1.17040
Z_{32}	$+2.6470$	$+11.7$	$+218.2$	$+0.87280$
Z_{33}	$+18.6009$			
Z_{41}	-16.8064	-11.4	-497.4	-1.79071
Z_{42}	-2.8853	$+4.4$	$+289.6$	$+1.04256$
Z_{43}	$+12.3932$	$+17.8$	-188.0	-0.67680
Z_{44}	$+21.2985$			
Z_{51}	-16.8934	-12.6	-708.8	-4.08311
Z_{52}	-4.6582	$+0.7$	$+252.4$	$+1.45433$
Z_{53}	$+7.7970$	$+12.9$	-156.7	-0.81259
Z_{54}	$+18.0760$	$+22.7$	$+192.9$	$+1.11110$
Z_{55}	$+25.6784$			
Z_{61}	-16.9010	-13.0	-1137.2	-16.37613
Z_{62}	-5.0133	-0.5	$+205.5$	$+2.96017$
Z_{63}	$+5.9979$	$+9.5$	-86.1	-1.24016
Z_{64}	$+15.7525$	$+19.2$	$+49.1$	$+0.70733$
Z_{65}	$+21.9884$	$+29.8$	-497.6	-7.16590
Z_{66}	$+33.1754$			
Z_{71}	-16.9015	-13.3	-1227.4	-63.63132
Z_{72}	-5.0500	-1.0	$+165.9$	$+8.60357$
Z_{73}	$+5.6193$	$+9.1$	-47.0	-2.44122
Z_{74}	$+14.7525$	$+17.8$	$+23.2$	$+1.20588$
Z_{75}	$+20.7688$	$+25.7$	-152.1	-7.87878
Z_{76}	$+28.6580$	$+39.5$	$+1431.7$	$+74.22189$
Z_{77}	$+43.1526$			

$$a_n = b^n (n - 1) /^2 \beta_n$$

TABLE V.

$b = 10$		$+\bar{\beta} = \text{Max.}$ $-\bar{\beta} = \text{Min.}$		
$a_n = 0, \beta_n = 0$		$\bar{z}$	$\bar{\beta}$	$\bar{a}$
Z_{21}	-0.0196	$+0.5$	-0.13	-0.00130×10^4
Z_{22}	$+1.0196$			
Z_{31}	-0.0196	$+0.4$	-0.11	-0.00048×10^6
Z_{32}	$+1.0163$	$+2.8$	$+0.75$	$+0.00300 \times 10^6$
Z_{33}	$+4.0033$			
Z_{41}	-0.0196	$+0.4$	-0.10	-0.00036×10^8
Z_{42}	$+1.0163$	$+2.7$	$+0.52$	$+0.00187 \times 10^8$
Z_{43}	$+4.0013$	$+7.7$	-3.43	-0.01235×10^8
Z_{44}	$+9.0019$			
Z_{51}	-0.0196	$+0.4$	-0.10	-0.00062×10^{10}
Z_{52}	$+1.0163$	$+2.7$	$+0.45$	$+0.00261 \times 10^{10}$
Z_{53}	$+4.0013$	$+7.0$	-1.96	-0.01134×10^{10}
Z_{54}	$+9.0005$	$+13.7$	$+15.84$	$+0.09129 \times 10^{10}$
Z_{55}	$+16.0014$			
Z_{61}	-0.0196	$+0.4$	-0.10	-0.00153×10^{12}
Z_{62}	$+1.0163$	$+2.6$	$+0.38$	$+0.00558 \times 10^{12}$
Z_{63}	$+4.0013$	$+6.9$	-1.41	-0.02041×10^{12}
Z_{64}	$+9.0005$	$+13.4$	$+7.20$	$+0.10364 \times 10^{12}$
Z_{65}	$+16.0003$	$+22.3$	-69.33	-0.99847×10^{12}
Z_{66}	$+25.0011$			
Z_{71}	-0.0196	$+0.4$	-0.10	-0.00547×10^{14}
Z_{72}	$+1.0163$	$+2.6$	$+0.35$	$+0.01850 \times 10^{14}$
Z_{73}	$+4.0013$	$+6.9$	-1.14	-0.05941×10^{14}
Z_{74}	$+9.0005$	$+13.2$	$+4.51$	$+0.23393 \times 10^{14}$
Z_{75}	$+16.0003$	$+21.9$	-26.29	-1.36291×10^{14}
Z_{76}	$+25.0002$	$+34.0$	$+1551.99$	$+80.45630 \times 10^{14}$
Z_{77}	$+36.0009$			

$$\bar{a} = b^n (n - 1) /^2 \bar{\beta}$$

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On Elliptic Modular Equations for Transformations of Orders 29, 31, 37.

By Arthur Berry.

§ 1. *Introduction.*

The problem of the transformation of elliptic functions gives rise to the problem of calculating modular equations corresponding to a transformation of any order; *i. e.*, of algebraic equations connecting some modular function of the ratio of the periods (τ) with the same function of the ratio of the transformed periods, which can in general be taken to be $n\tau$, or τ/n, where n is the order of the transformation. Jacobi[*] used the modular function $u \equiv \sqrt[4]{k}$, for which Hermite introduced the notation $\phi(\tau)$, and computed the modular equations for $n = 3, 5$, as rational equations between u and the transformed modulus $v \equiv \sqrt[4]{\lambda}$. Sohncke[†] subsequently dealt with the cases of $n = 7, 11, 13, 17, 19$. As far as I know, no higher cases have been worked out in this form, but a large number of modular equations have been worked out in terms of various irrational functions of k, λ conjointly. The two most extensive sets of equations have been given by Schroeter,[‡] who used irrational functions differing from order to order and computed the equations for prime orders up to 31, as well as for certain non-prime orders, and by R. Russell,[§] who worked systematically with the functions $k\lambda$, $k'\lambda'$ and their square roots and fourth roots and obtained equations for prime orders up to 59, with the exception of 41 (for which case his work is not quite finished) and 37, as well as for several non-prime orders and for various higher prime orders. A number of irrational modular equations were also given a little earlier by E. W. Fiedler[||] in his inaugural dissertation.

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[*] *Fundamenta Nova*, §§ 13, 15.

[†] *Crelle's Journal*, vol. 16 (1836).

[‡] *De Aequationibus Modularibus*, Königsberg, 1854, and *Crelle's Journal*, vol. 58 (1860).

[§] *Proceedings of the London Mathematical Society*, Series I, vol. 19 (1889), vol. 21 (1891).

[||] Ueber eine besondere Classe irrationaler Modulargleichungen der elliptischen Functionen. Zürich, 1885 also in Wolf's *Zeitschrift*, vol. 30.

In 1858 Hermite[*] introduced a new modular function $\chi(\tau) \equiv \sqrt[12]{(kk')}$. The corresponding modular equations were given by Schläfli[†] for prime orders up to 19, and the equations frequently bear his name; the case of $n = 23$ was given by Weber[‡] as an illustration of the theory of complex multiplication.

Klein, in a well-known paper,[§] considered the modular equations formed with the absolute invariant J, and gave explicit formulae for J and the transformed function J′ as rational functions of a parameter, in the cases $n = 3, 4, 5, 7, 13$. In the same volume Gierster worked out the remaining cases in which the modular equation is of deficiency zero; viz., the cases of $n = 6, 8, 9, 10, 12, 16, 18, 25$.

A comparison of the modular equations formed respectively with J, u, χ shews that the first are functionally much the simplest and the last the most complicated. For example, the deficiency (genus) of the u, v equation for $n = 3$ is already 7, and for $n = 5$ is 15; while the deficiency of Schläfli's equation for $n = 5$ is 23. But the order of numerical simplicity is the reverse. The modular equation in J for $n = 3$, which I believe to be the highest which has been explicitly calculated,[||] consists of 17 terms and contains the numerical coefficient $2^{15}.\,5^6.\,22973$; whereas for the higher case of $n = 5$ the u, v equation consists of only 6 terms with no coefficient greater than 5, and the Schläfli equation (with a slight numerical modification) assumes the extremely simple form

$$x^6 + y^6 - xy + 4x^5y^5 = 0.$$

The object of this paper is to establish what I believe to be a new property of the Schläfli modular equations (§3), and to compute the equations for the cases $n = 29, 31, 37$, the last case being one for which, as far as I know, no modular equation in any form has been computed.

§ 2. *The Modular Function* $x(\tau)$.

I find it convenient for numerical purposes slightly to modify Hermite's function and to work instead with

$$2^{-1/6}.\sqrt[12]{(kk')} \equiv 2^{-1/6}\chi(\tau) \equiv q^{1/24} / \prod_1^\infty (1 + q^{2m-1}).$$

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[*] Sur la résolution de l'équation du quatrième degré, *Comptes Rendus*, vol. 46 (1858), reprinted with other papers in the pamphlet, Sur la théorie des équations modulaires et la résolution de l'équation du cinquième degré. Paris (1859).

[†] *Crelle's Journal*, vol. 72 (1870).

[‡] In his book *Elliptische Functionen und Algebraische Zahlen*, § 99.

[§] Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades, *Mathematische Annalen*, vol. 14 (1879).

[||] It is given in a slightly different notation by H. J. S. Smith, *Proceedings of the London Mathematical Society*, vol. 9 (1878), and *Collected Mathematical Papers*, vol. 2, p. 242.

This is the reciprocal of Weber's function $f(\omega)$; I denote it for convenience by $x(\tau)$, and denote the corresponding transformed function $x(n\tau)$ by y, so that the required modular equation appears as an equation in x and y.

It is known that $x(-{}^1/\tau) = x(\tau)$; $x(2+\tau) = \varepsilon x(\tau)$, where $\varepsilon = e^{2i\pi/24}$. The modular equation for a transformation of prime order n, greater than 3, is of order $n+1$ in either x or y and symmetrical in them. The $n+1$ values of y are $y_\infty \equiv x(n\tau)$, and $y_r \equiv x((48r+\tau)/n)$ $(r = 0, 1 \ldots . n-1)$. When x vanishes all the values of y vanish also, and the corresponding approximations are given by $y^n = x$, and $y = x^n$. Also the modular equation is unaffected if x is replaced by εx, y by $\varepsilon^n y$; since a term y^{n+1} must occur, it at once follows that if a term $x^\alpha y^\beta$ occurs $\alpha + n\beta \equiv n+1$, *mod.* 48. The high modulus of this congruence explains why the number of terms is small compared with the number in the corresponding u, v or J, J$'$ equations. Further by means of the quadratic transformation $\{\tau, (1+\tau)/(1-\tau)\}$ it can be shewn that the modular equation is unaffected if in it we write $2^{-1/2}x^{-1}$, $\left(\frac{2}{n}\right)2^{-1/2}y^{-1}$ for x, y respectively,* where $\left(\frac{2}{n}\right)$ is Legendre's symbol and denotes $+1$ or -1 as n is of the form $8m \pm 1$ or $8m \pm 3$. We shall refer to this property briefly as *reciprocity.*

These properties enable us to write down the possible terms in the modular equation, and to assign coefficients to four of them; viz., we always have the four terms

$$x^{n+1} + y^{n+1} - xy - \left(\frac{2}{n}\right)2^{(n-1)/2}x^n y^n;$$

moreover, a large number of the remaining coefficients are not independent, since the coefficient of $x^\alpha y^\beta$ is equal to that of $x^\beta y^\alpha$, while that of $x^{n+1-\alpha} y^{n+1-\beta}$ can be at once derived from either by reciprocity. Lastly, the coefficients are all integers, and save for the four "known" terms just written down, they are all multiples of n.

§ 3. *The Branch Places of the (x, y) Modular Equation.*

The approximations at the origin and at infinity given in the last paragraph show that y, regarded as a function of x, has at $x=0$ and $x=\infty$ n branches which are cyclically permuted when x describes a circuit round the place, and one distinct branch; in other words, the Riemann surface has at each of the places $x=0$, $x=\infty$ a winding-point of order n and an isolated sheet.

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* These properties and their proofs are all to be found in Weber's book.

The remaining branch places are given by $x^{24} = 2^{-6}$, or $\tau = i + 2p$, where p is an integer. To prove this and to investigate the nature of the branching let us first consider $x = 2^{-1/4}$, $\tau = i$.

The substitution T or $(\tau, -1/\tau)$ applied to τ leaves $x(\tau)$ unchanged, and a slight modification of a familiar process shows that this substitution interchanges the roots y_0, y_∞ of the modular equation and also the roots y_r, y_s, where

$$1 + 48^2 . r . s \equiv 0, \textit{ mod. } n.$$

Thus y_∞ becomes

$$x(nT\tau) = x(-n/\tau) = x(\tau/n) = y_0.$$

Further, y_r becomes

$$x\left(\frac{48r + T\tau}{n}\right) = x\left(\frac{-1 + 48r\tau}{n\tau}\right) = x\left(\frac{-1 - 48^2 . r . s + 48r(48s + \tau)}{n\tau}\right).$$

If now we choose s a positive integer less than n to satisfy the congruence $1 + 48^2 rs \equiv 0$, *mod.* n, say $1 + 48^2 rs = cn$, where c is an integer, we have

$$x\left(\frac{-c + 48r . \tau_s}{-48s + n\tau_s}\right),$$

where τ_s is written for $\tau(48s + \tau)/n$. This expression is of the form

$$x(c + d\tau_s/a + b\tau_s),$$

where a, b, c, d are integers such that $ad - bc = 1$; also b, c are odd, a, d are even, and $b - c = n - \dfrac{1 + 48^2 rs}{n} = \dfrac{n^2 - 1 - 48^2 rs}{n} \equiv 0$, *mod.* 48, since n is a prime number other than 2 or 3. Hermite's formulæ for linear transformation of the function $x(\tau)$, or $\chi(\tau)$,* show that our expression reduces to $x(\tau_s)$ or y_s. Now the congruence $1 + 48^2 rs \equiv 0$, *mod.* n, always admits of a solution for any value of r from 1 to $n - 1$, since n is prime to 48; also the congruence is symmetrical in r, s; hence in general the substitution T permutes the two roots y_r, y_s. The exceptional case is when r coincides with s, in which case the corresponding root is unaffected. The congruence $1 + 48^2 r^2 \equiv 0$, *mod.* n, evidently admits of solutions under the same conditions as the congruence $1 + r^2 \equiv 0$, and therefore, by a well-known result of the theory of numbers,† there are two solutions or none as n is of the form $4p + 1$ or $4p - 1$. Hence the substitution T applied to τ interchanges in pairs $n - 1$ or $n + 1$ of the roots of the modular equation, while there are two or no isolated roots. Since $Ti = i$ and $x(i) = 2^{-1/4}$, we thus see that:

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* See for example Tannery and Molk's *Théorie des Fonctions Elliptiques*, vol. 2, Table XLVI.

† Mathews' *Theory of Numbers*, § 37.

When $x = 2^{-1/4}$, the Riemann surface has $(n-1)/2$ or $(n+1)/2$ simple winding-points, at each of which two roots are interchanged, while 2 or 0 sheets are distinct, according as n is of the form $4p+1$ or $4p-1$.

Further, it can be shown that the two isolated values of y, when they occur (i. e., in the case $n = 4p+1$), are always equal to the value of x multiplied by ± 1 or $\pm i$. This can be established by using the quadratic transformation $\left(\tau, \frac{1+\tau}{1-\tau}\right)$, of which $\tau = i$ is a fixed point. This substitution converts x into $2^{-1/2}x^{-1}$, $x(n\tau)$ into $\left(\frac{2}{n}\right) 2^{-1/2}/x(n\tau)$.* Now if $\tau' = (1+\tau)/(1-\tau)$,

$$x\{(\tau' + 48r)/n\} = x\left(\frac{-n}{\tau' + 48r}\right) = x\left\{\frac{n(-1+\tau)}{1 + 48r + (1-48r)\tau}\right\}.$$

Choose r so that $1 + 48^2 r^2 \equiv 0$, *mod.* n, or $1 + 48^2 r^2 = nn'$, where n' is an integer. Then our function becomes $x\left\{\frac{-(1+48r)+\tau_r}{n' + (1-48r)\tau_r}\right\}$, (where $\tau_r = (\tau + 48r)/n$), which is $x\left(\frac{1+U\tau_r}{1-U\tau_r}\right)$, where $U\tau \equiv \frac{c + d\tau}{a + b\tau}$ and $2a = -1 - 48r + n'$, $2b = 1 - 48r + n$, $2c = -1 - 48r + n'$, $2d = -1 + 48r + n$. Here a, b, c, d are integers satisfying the relation $ad - bc = 1$, and it is easy to verify that $(a+d)(abd - c) \equiv 24$ or 0, *mod.* 48, according as n is of the form $8p-3$ or $8p+1$. Hence $x\left(\frac{1+U\tau}{1-U\tau}\right) = 2^{-1/2}/x(U\tau_r) = \left(\frac{2}{n}\right) \cdot 2^{-1/2}/x(\tau_r)$ by Hermite's formulae for the linear transformation of $x(\tau)$.

Now if $\tau = i$, $\tau' = i$ also, so that we have proved that for this value of τ, $y_r = \left(\frac{2}{n}\right) \cdot 2^{-1/2}/y_r$, where r is either root of the congruence $1 + 48^2 r^2 \equiv 0$, *mod.* n. Thus, if $n = 8p-3$, $y_r = \pm i2^{-1/4}$, and, if $n = 8p+1$, $y_r = \pm 2^{-1/4}$. Moreover, the two isolated roots of the equation, corresponding to the two roots of the congruence, are of the form $x(\pm\alpha + i\beta)$, so that by a known result one is always conjugate to the other. Hence in one case the two isolated roots are $i2^{-1/4}$ and $-i2^{-1/4}$ respectively, and in the other they are both $2^{-1/4}$ or both $-2^{-1/4}$.

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* Weber, *loc. cit.*, § 76.

I have not succeeded in finding any simple criterion distinguishing these last two cases. It is however quite easy in any concrete case to reduce $y_r \equiv x\,\{(i+48r)/n\}$ by continual application of the substitutions T or $(\tau, -1/\tau)$ and S^2 or $(\tau, 2+\tau)$ to the form $\varepsilon^\lambda x(i)$. For example, if $n = 113$ the roots of the congruence are $r = 35$ or 78; now using in succession the identities $98^2 + 1 = 113.85$, $72^2 + 1 = 85.61$, $50^2 + 1 = 61.41$, $32^2 + 1 = 41.25$, $18^2 + 1 = 25.13$, $8^2 + 1 = 13.5$, we obtain

$$\begin{aligned} x\left(\frac{i+48.35}{113}\right) &= \varepsilon^7 x\left(\frac{i+98}{113}\right) = \varepsilon^7 x\left(\frac{85}{-i+98}\right) = \varepsilon^7 x\left(\frac{i-98}{85}\right) \\ &= \varepsilon^6 x\left(\frac{i+72}{85}\right) = \varepsilon^6 x\left(\frac{61}{-i+72}\right) = \varepsilon^6 x\left(\frac{i-72}{61}\right) = \varepsilon^5 x\left(\frac{i+50}{61}\right) \\ &= \varepsilon^5 x\left(\frac{41}{-i+50}\right) = \varepsilon^5 x\left(\frac{i-50}{41}\right) = \varepsilon^4 x\left(\frac{i+32}{41}\right) = \varepsilon^4 x\left(\frac{25}{-i+32}\right) \\ &= \varepsilon^4 x\left(\frac{i-32}{25}\right) = \varepsilon^3 x\left(\frac{i+18}{25}\right) = \varepsilon^3 x\left(\frac{13}{-i+18}\right) = \varepsilon^3 x\left(\frac{i-18}{13}\right) \\ &= \varepsilon^2 x\left(\frac{i+8}{13}\right) = \varepsilon^2 x\left(\frac{5}{-i+8}\right) = \varepsilon^2 x\left(\frac{i-8}{5}\right) = \varepsilon\, x\left(\frac{i+2}{5}\right) \\ &= \varepsilon\, x\left(\frac{1}{-i+2}\right) = x(i) = 2^{-1/4}, \text{ so that in this case } \lambda = 24. \end{aligned}$$

I have tested in this way all the primes of the forms $4p + 1$ up to 113. In all those of the form $8p - 3$, λ is ± 6 in accordance with the theory; for $n = 17, 41, 97$, λ is 12; and for $n = 73, 89, 113$, λ is 24.

It is obvious that exactly like properties hold at each of the 24 points given by $x^{24} = 2^{-6}$, and it is easily seen that there are no other branch places, since a branch place is clearly a fixed point of a substitution belonging to that subgroup of the modular group $\left(\tau, \dfrac{c+d\tau}{a+b\tau}\right)$ which leaves $x(\tau)$ unaltered. From the known properties of the modular group it readily follows that of the three fundamental singular points $\tau = \infty$, $\tau = i$, $\tau = \dfrac{-1+i\sqrt{3}}{2}$ of the modular group, the last gives rise to no fixed point of a substitution belonging to our subgroup, while the first two give the branch places 0, ∞, $\sqrt[24]{2^{-6}}$ already considered.

§ 4. *The Modular Equation for* $n = 29$.

Using the properties quoted in § 2, we have for the equation

$$
\begin{aligned}
&- xy\,(1 - 2^{14}x^{28}y^{28}) \\
&+ 29ax^5y^5\,(1 - 2^{10}x^{20}y^{20}) \\
&+ 29b\,(x^{10}y^4 + x^4y^{10})\,(1 + 2^8x^{16}y^{16}) \\
&+ 29\,(c_1x^{15}y^3 + c_2x^9y^9 + c_1x^3y^{15})\,(1 - 2^6x^{12}y^{12}) \\
&+ 29\,(d_1x^{20}y^2 + d_2x^{14}y^8 + d_2x^8y^{14} + d_1x^2y^{20})\,(1 + 2^4x^8y^8) \\
&+ 29\,(e_1x^{25}y + e_2x^{19}y^7 + e_3x^{13}y^{13} + e_2x^7y^{19} + e_1xy^{25})\,(1 - 2^2x^4y^4) \\
&+ x^{30} + y^{30} + 29\,(f_1x^{24}y^6 + f_2x^{18}y^{12} + f_2x^{12}y^{18} + f_1x^6y^{24}) = 0,
\end{aligned}
$$

where $a, b, c_1, c_2, d_1, d_2, e_1, e_2, e_3, f_1, f_2$ are eleven integers which have to be determined.

Substituting for x, y the q functions, $q^{1/24}/(1+q)\,(1+q^3)\ldots.$ and $q^{29/24}/(1+q^{29})\,(1+q^{3.29})\ldots.$, we have an identity in q. It is convenient to divide by xy so as to get rid of fractional powers, and then to multiply by a power of $(1+q)\,(1+q^3)\ldots.$, so as to avoid as far as possible high indices in the binomials. If we retain only terms up to q^5 we have 5 equations giving in succession $e_1 = 1$, $d_1 = 9$, $c_1 = 27$, $b = 23$, $a = 7$.

We next consider the factors of the modular equation corresponding to the case of complex multiplication given by $y = x$. Making this substitution and writing for brevity z for x^4, the modular equation reduces to

$$
\begin{aligned}
f(z) \equiv -1 + 29az^2 + 58bz^3 + 29Cz^4 + 29Dz^5 + 29Ez^6 + (2 + 29F)z^7 \\
- 2^2.\,29\,.\,E\,.\,z^8 + \ldots. + 2^{14}z^{14} = 0,
\end{aligned}
$$

where $C = 2c_1 + c_2$, $D = 2\,(d_1 + d_2)$, $E = 2e_1 + 2e_2 + e_3$, $F = 2\,(f_1 + f_2)$, and the coefficients not written down can be at once supplied if wanted from reciprocity.

1. Corresponding to the expression of 29 as the quadratic form $4^2 + 13$, we have

$$
x\left(\frac{4 + i\sqrt{13}}{29}\right) = x\left(\frac{1}{4 - i\sqrt{13}}\right) = x(-4 + i\sqrt{13}) = \varepsilon^{-1}x(4 + i\sqrt{13}),
$$

and it readily follows that

$$
x\left(\frac{48 + (6 + i\sqrt{13})}{29}\right) = \varepsilon x\left(\frac{-4 + i\sqrt{13}}{29}\right) = \varepsilon x\,(4 + i\sqrt{13}) = x\,(6 + i\sqrt{13}.)
$$

Thus $y_1 \equiv x\left(\frac{48 + \tau}{29}\right) = x\,(\tau)$, when $\tau = 6 + i\sqrt{13}$, and similarly $y_6 = x\,(\tau)$.

Thus $z = x^4(6 + i\sqrt{13})$ is a repeated root of $f(z) = 0$. But the modular equation for $n = 13$ gives $x^4(i\sqrt{13}) = \dfrac{\sqrt{13} - 3}{4}$, so that $z = \dfrac{-\sqrt{13} + 3}{4}$; rationalizing this we have $4z^2 - 6z - 1$ as a repeated factor of $f(z)$.

2. Similarly from the expression of 29 as $1^2 + 28$, we deduce

$$x\left(\frac{28 + i\sqrt{28}}{2\,.\,29}\right) = x\left(\frac{-14 + i\sqrt{7}}{7}\right) = \varepsilon^{-1} x(i\sqrt{7}),$$

whence

$$y_2 = x\left(\frac{2\,.\,48 + (6 + i\sqrt{7})}{29}\right) = \varepsilon^2 x\left(\frac{-14 + i\sqrt{7}}{29}\right) = \varepsilon^3 x(i\sqrt{7}) = x(6 + i\sqrt{7}),$$

and similarly $y_5 = x$ for the same value of the argument. From the modular equation for $n = 7$ we have $x(i\sqrt{7}) = 1/\sqrt{2}$, so that $z \equiv x^4(6 + i\sqrt{7}) = -1/4$; we have therefore the repeated factor $4z - 1$, and by reciprocity we have associated with this the factor $z + 1$, and therefore the repeated factor $4z^2 + 3z - 1$.

3. We have obviously $x\left(\dfrac{i\sqrt{29}}{29}\right) = x(i\sqrt{29})$, *i.e.* $y_0 = x$, when $\tau = i\sqrt{29}$; and there are also associated with this two values $\dfrac{\pm 4 + i\sqrt{29}}{3}$, for which we have respectively $y_4 = x$ and $y_{25} = x$. We might quote the values of $x(i\sqrt{29})$ from Weber's table,[*] but I purposely use only results connected with modular equations of lower order than the one under discussion.

We have now accounted for all the factors of $f(z)$, so that

$$-f(z) \equiv (1 + 6z - 4z^2)^2 (1 + 3z - 4z^2)^2 (1 + \alpha z + \beta z^2 + \gamma z^3 - 2^2 \beta z^4 + 2^4 \alpha z^5 - 2^6 z^6),$$

where the last factor corresponds to the values just discussed.

Equating coefficients of z, z^2, z^3, we have

$$\begin{aligned} 0 &= 18 + \alpha \\ -29a &= 101 + 18\alpha + \beta \\ -58b &= 108 + 101\alpha + 18\beta + \gamma, \end{aligned}$$

whence, a and b being known, $\alpha = -18$, $\beta = 20$, $\gamma = 16$. We have thus shown that $x(i\sqrt{29})$ satisfies the sextic equation

$$1 - 18z + 20z^2 + 16z^3 - 2^2 . 20z^4 - 2^4 . 18z^5 - 2^6 z^6 = 0,$$

which agrees with Weber's equation.

---

[*] *Loc. cit.*, pp. 499–504.

Equating coefficients of z^4, z^5, z^6, z^7, we now deduce

$$C = 8, \quad D = -376, \quad E = 432, \quad F = 2{,}966,$$

whence

$$c_2 = -46, \quad d_2 = -197, \quad 2e_2 + e_3 = 430, \quad f_1 + f_2 = 1{,}483.$$

We now want one more equation connecting e_2 and e_3 and one connecting f_1 and f_2. These might be obtained from the complex theory arising from $y = \varepsilon x$ or $y = \varepsilon^2 x$, and I originally obtained the missing coefficients by this method; but it is perhaps a little simpler to use the property given by § 3.

When $\tau = i$, or $x = 2^{-1/4}$, two of the roots of the modular equation are isolated and equal to $\pm i2^{-1/4}$; the rest are equal in pairs. On substituting $x = 2^{-1/4}$ and, for convenience, $y = 2^{-1/4}\eta$, we have consequently

$$\begin{aligned} 1 &+ \eta\,(2\,.\,29\,.\,e_1 - 2^7) + \eta^2\, 2^2.\,29\,.\,d_1 + \eta^3.\,2^3.\,29\,.\,c_1 + \eta^4.\,2^4.\,29\,.\,b \\ &+ \eta^5.\,29\,(2^5 a - 2e_1) + \eta^6.\,29\,.\,f_1 + \eta^7.\,2\,.\,29\,.\,e_2 + \eta^8.\,2^2.\,29\,.\,d_2 \\ &+ \eta^9.\,2^3.\,29\,.\,c_2 + \eta^{10}.\,29\,.\,(2^2 d_1 + 2^4 b) - \eta^{11}\,2\,.\,29\,.\,e_2 + \eta^{12}.\,29\,.\,f_2 \\ &+ \eta^{13}.\,2\,.\,29\,.\,e_3 + \eta^{14}.\,2^2.\,29\,.\,d_2 + \eta^{15}.\,0 + \eta^{16}.\,2^2.\,29\;d_2 - \eta^{17}.\,2\,.\,29\,.\,e_3 \ldots\ldots \\ &+ \eta^{29}\,(-2\,.\,29\,.\,e_1 + 2^7) + \eta^{30} = (1 + \eta^2)\,(1 + k_1\eta + k_2\eta^2 + k_3\eta^3 \ldots\ldots \\ &+ k_7\eta^7 - k_6\eta^8 + k_5\eta^9 - k_4\eta^{10} + k_3\eta^{11} - k_2\eta^{12} + k_1\eta^{13} - \eta^{14})^2. \end{aligned}$$

Equating in this identity the coefficients of $\eta, \eta^2, \ldots\ldots, \eta^5$, we obtain successively $k_1 = -35$, $k_2 = -91$, $k_3 = -18$, $k_4 = 44$, $k_5 = -46$. Equating coefficients of η^8, η^9, we obtain $k_6 = 100$ and $k_7 = +54$. Equating coefficients of η^6, η^7, we then obtain $f_1 = 185$, $e_2 = 0$, whence, from the values already found, $e_3 = 430$, $f_2 = 1298 = 2\,.\,11\,.\,59$. Thus all the coefficients of the modular equation have been obtained. If in our identity we further equate coefficients of η^{10}, η^{11}, η^{12}, η^{13}, η^{14}, we obtain 5 new relations between the coefficients which may serve as equations of verification.

Collecting the results, we see that the modular equation for $n = 29$ is of the form written at the beginning of this paragraph, with the numerical coefficients:

$$\begin{aligned} &a = 7, \quad b = 23, \quad c_1 = 27, \quad c_2 = -46, \quad d_1 = 9, \quad d_2 = -197, \\ &\qquad e_1 = 1, \quad e_2 = 0, \quad e_3 = 430, \quad f_1 = 185, \quad f_2 = 1{,}298 = 2\,.\,11\,.\,59. \end{aligned}$$

§ 5. *The Modular Equation for* $n = 31$.

As before, the equation is

$$\begin{aligned}
&-xy\,(1 + 2^{15}x^{30}y^{30}) \\
&+ 31\,.\,a\,.\,x^4y^4\,(1 + 2^{12}x^{24}y^{24}) \\
&+ 31\,(b_1x^{11}y^3 + b_2x^7y^7 + b_1x^3y^{11})\,(1 + 2^9x^{18}y^{18}) \\
&+ 31\,(c_1x^{18}y^2 + c_2x^{14}y^6 + c_3x^{10}y^{10} + c_2x^6y^{14} + c_1x^2y^{18})\,(1 + 2^6x^{12}y^{12}) \\
&+ 31\,(d_1x^{25}y + d_2x^{21}y^5 + d_3x^{17}y^9 + d_4x^{13}y^{13} + d_3x^9y^{17} + d_2x^5y^{21} \\
&\qquad\qquad + d_1xy^{25})\,(1 + 2^3x^6y^6) \\
&+ x^{32} + y^{32} + 31\,(e_1x^{28}y^4 + e_2x^{24}y^8 + e_3x^{20}y^{12} + e_4x^{16}y^{16} + e_3x^{12}y^{20} \\
&\qquad\qquad + e_2x^8y^{24} + e_1x^4y^{28}) = 0,
\end{aligned}$$

where a, b_1, b_2, c_1, c_2, c_3, d_1, d_2, d_3, d_4, e_1, e_2, e_3, e_4 are fourteen integers to be determined.

If we substitute, as before, the q-products for x, y, divide out by xy, multiply by $\{(1+q)\,(1+q^3)\ldots.\}^{20}$, and expand as far as q^7, we find successively

$$d_1 = 1, \quad c_1 = 8, \quad b_1 = 13, \quad a = 3, \quad e_1 = 25, \quad d_2 = -256, \quad c_2 = -205.$$

Let us consider next the complex multiplications of the type $y = \varepsilon^2 x$, which arise from the resolutions $31 = 2^2 + 27$, $31 = 4^2 + 15$. We verify at once that $x\{(-1 + i\sqrt{27})/31\} = \varepsilon^2 x(-2 + i\sqrt{27})$, so that $\varepsilon^{-1}x(i\sqrt{27})$ satisfies the equation $y = \varepsilon^2 x$. From the known modular equation for $n = 3$ we find that $z = x^6(i\sqrt{27})$ satisfies the cubic equation $1 - 60z + 48z^2 - 64z^3 = 0$.

Similarly $y_9 = x\{(9\,.\,48 + 6 + i\sqrt{15})/31\} = \varepsilon^2 x(6 + i\sqrt{15})$, so that $x(6 + i\sqrt{15}) = \varepsilon^3 x(i\sqrt{15})$ is also a solution of $y = \varepsilon^2 x$. From the modular equation of order 19 we have $x^6(i\sqrt{19}) = \dfrac{3 - \sqrt{5}}{24}$, so that $z = x^6(i\sqrt{19})$ satisfies the quadratic equation $1 - 24z + 64z^2 = 0$.

If in the modular equation we put (to avoid imaginaries) $x = \varepsilon\xi$, $y = \varepsilon^3\xi$, we get an equation of degree 10 in $z \equiv \xi^6$, satisfied by $x^6(i\sqrt{19})$ and $x^6(i\sqrt{27})$, of which we have just found a quadratic and a cubic factor. Using reciprocity we obtain the remaining factors.

We thus have the identity

$$\begin{aligned}
&1 - 31az - 31B'z^2 - 31C'z^3 - 31D'z^4 + (1 - 31E')z^5 \ldots. + 2^{15}z^{10} \\
&\quad \equiv (1 - 60z + 48z^2 - 64z^3)\,(1 - 6z + 60z^2 - 8z^3)\,(1 - 24z + 64z^2)\,(1 - 3z + z^2),
\end{aligned}$$

where

$$\begin{aligned}
B' &= -b_1 + b_2, & C' &= -c_1 - c_2 + c_3, \\
D' &= 2d_1 - d_2 - d_3 + d_4, & E' &= 2e_1 - e_2 - e_3 + e_4.
\end{aligned}$$

Equating coefficients, we obtain

$a = 3$ (verifying the previous result),

$$B' = -77, \quad C' = 834, \quad D' = -6{,}100, \quad E' = 24{,}295;$$

whence

$$b_2 = -64, \quad c_3 = 637, \quad d_3 - d_4 = 6{,}358, \quad e_2 + e_3 - e_4 = -24{,}245.$$

We can now obtain two more equations from the complex multiplication of the type $y = x$. It is easily verified that if

$$\tau = \frac{1 + i\sqrt{3}}{2}, \text{ each of } y_{27} \equiv x\left(\frac{27 \,.\, 48 + \tau}{31}\right) \text{ and } y_7 \equiv x\left(\frac{7 \,.\, 48 + \tau}{31}\right)$$

is equal to $x(\tau)$; so that $x\left(\frac{1 + i\sqrt{3}}{2}\right)$ is a repeated root of $y = x$, and this quantity is well known to be $2^{-1/6}$. We have also obviously

$$x\left(\frac{i\sqrt{31}}{31}\right) = x(i\sqrt{31}),$$

so that $x(i\sqrt{31})$ is another root of our equation, but we do not assume this quantity to be known.

Using reciprocity, we now have, on putting $y = x$ in the modular equation, and then writing z for x^6,

$$\begin{aligned}(1 - 31az - 31\ Bz^2 - 31\ Cz^3 - 31\ Dz^4 - (2 + 31E)\,z^5 \ldots + 2^{15} z^{10})& \\ \equiv (1 - 2z)^2 (1 - 4z)^2 (1 + \alpha z + \beta z^2 + \gamma z^3 + 2^3 \beta z^4 + 2^6 \alpha z^5 + 2^9 z^6)&,\end{aligned}$$

where α, β, γ are at present unknown, and

$$B = 2b_1 + b_2 = -38, \qquad C = 2c_1 + 2c_2 + c_3 = 243,$$
$$D = 2d_1 + 2d_2 + 2d_3 + d_4, \qquad E = 2e_1 + 2e_2 + 2e_3 + e_4.$$

Equating coefficients of z, z^2, z^3, we have

$$\alpha - 12 = -31 \,.\, a, \qquad \beta - 12\alpha + 52 = 31 \,.\, 38,$$
$$\gamma - 12\beta + 52\alpha - 96 = -31 \,.\, 241;$$

whence

$$\alpha = -81, \qquad \beta = 154, \qquad \gamma = -1{,}377.$$

Equating coefficients of z^4, z^5, we deduce

$$D = -1{,}084, \qquad E = 3{,}598;$$

whence

$d_3 = 1{,}928 = 2^3 . 241$, $d_4 = -4{,}430$, $e_4 = 17{,}346 = 2 \,.\, 3 \,.\, 7^2 . 59$, $e_2 + e_3 = -6{,}899$.

We have shown incidentally that $z = x^6(i\sqrt{31})$ satisfies the equation

$$1 - 81z + 154z^2 - 1{,}377z^3 + 2^3 .\, 154z^4 - 2^6 .\, 81z^5 + 2^9 z^6 = 0,$$

agreeing with Weber's result.

We still want one more equation connecting e_2 and e_3. This can be obtained from complex multiplication of the type $y = \varepsilon x$. Corresponding to the resolution $31 = 5^2 + 6$, we easily find that if

$$\tau = -1 + i\sqrt{6}, \quad y_4 \equiv x\left(\frac{\tau + 4 \,.\, 48}{31}\right) = \varepsilon x(\tau),$$

so that $x(-1 + i\sqrt{6})$ is a solution of $y = \varepsilon x$. But from the modular equation for $n = 7$ we have $x^6(-1 + i\sqrt{6}) = \varepsilon^{-3}(2 - \sqrt{2})/4$. If therefore we put $y = \varepsilon x$ and then $x^6 = \varepsilon^{-3} z$, the resulting equation in z is satisfied by $(2 - \sqrt{2})/4$. The equation is

$$\begin{aligned} -1 - 2^{15} z^{10} + 31 a\, z(1 + 2^{12} z^8) + 31 B'' z^2 (1 + 2^9 z^6) + 31\, C'' z^3 (1 + 2^6 z^4) \\ + 31 D'' z^4 (1 + 2^3 z^2) + (-1 + 31 E'') z^5 = 0, \end{aligned}$$

where

$$B'' = b_1 + b_2, \qquad C'' = -c_1 + c_2 + c_3,$$
$$D'' = -2d_1 - d_2 + d_3 + d_4, \qquad E'' = -2e_1 - e_2 + e_3 + e_4.$$

Putting $z = (2 - \sqrt{2})/4$ and substituting the known values of a, B'', C'', D'', we obtain

$$E'' = 7{,}903, \quad \text{whence} \quad e_2 - e_3 = 9{,}393,$$

and, since we know $e_2 + e_3$,

$$e_2 = 1{,}247, \qquad e_3 = -8{,}146.$$

Summing up the results, we see that the modular equation for $n = 31$ is of the form written at the beginning of this paragraph, with the numerical coefficients:

$$a = 3, \quad b_1 = 13, \quad b_2 = -64, \quad c_1 = 8, \quad c_2 = -205, \quad c_3 = 637 = 7^2 .\, 13,$$
$$d_1 = 1, \quad d_2 = -256, \quad d_3 = 1{,}928 = 2^3 .\, 241, \quad d_4 = -4{,}430, \quad e_1 = 25,$$
$$e_2 = 1{,}247 = 29 \,.\, 43, \qquad e_3 = -8{,}146, \qquad e_4 = 17{,}346 = 2 \,.\, 3 \,.\, 7^2 .\, 59.$$

I have checked the accuracy of these results by carrying the expansions as far as q^{13}, and further by verifying that when $\tau = i$, $x = 2^{-1/4}$, the roots of the modular equation are all equal in pairs.

§ 6. *The Modular Equation for $n = 37$.*

The equation is

$$\begin{aligned}
&-xy\,(1-2^{18}x^{36}y^{36})\\
&+37\,(a_1x^{12}y^2+a_2x^{10}y^4+a_3x^8y^6+a_3x^6y^8+a_2x^4y^{10}+a_1x^2y^{12})\,(1+2^{12}x^{24}y^{24})\\
&+37\,(b_1x^{25}y+b_2x^{23}y^3+b_3x^{21}y^5+b_4x^{19}y^7+b_5x^{17}y^9+b_6x^{15}y^{11}+b_7x^{13}y^{13}\\
&\quad+b_6x^{11}y^{15}+b_5x^9y^{17}+b_4x^7y^{19}+b_3x^5y^{21}+b_2x^3y^{23}+b_1xy^{25})\,(1-2^6x^{12}y^{12})\\
&+x^{38}+y^{38}+37\,(c_1x^{36}y^2+c_2x^{34}y^4+\dots+c_9x^{20}y^{18}+c_9x^{18}y^{20}+\dots\\
&\qquad\qquad+c_2x^4y^{34}+c_1x^2y^{36})=0,
\end{aligned}$$

where $a_1, a_2, a_3, b_1, \dots, b_7, c_1, \dots, c_9$ are nineteen integers to be determined.

If we substitute, as before, q-products for x, y, divide by xy, multiply by $\{(1+q)(1+q^3)\dots\}^{22}$, and expand as far as q^{12}, we get in succession

$$\begin{array}{llll}
b_1 = 1, & a_1 = 5, & c_1 = 3, & b_2 = -108,\\
a_2 = 51, & c_2 = 119, & b_3 = -333, & a_3 = 133,\\
c_3 = 2{,}073, & b_4 = 540, & c_4 = 16{,}558, & b_5 = 4{,}806,
\end{array}$$

and have also one superfluous equation serving as a verification.

It is now possible to complete the calculation by means of the property of § 3. When $\tau = i$, $x = 2^{-1/4}$, we have two isolated roots, viz. $y = \pm i2^{-1/4}$, of the modular equation, and the other roots are equal in pairs. If therefore we put $x = 2^{-1/4}$, $y = 2^{-1/4}\eta$, the left-hand side of the modular equation reduces to $1+\eta^2$ multiplied by the square of a polynomial in η of degree 18. By reciprocity only 9 coefficients in this polynomial are independent, and by means of the known coefficients $a_1, a_2, a_3, b_1, \dots, b_5, c_1, \dots, c_4$, they can readily be computed by equating coefficients, or by extracting a square root. The left-hand side of the modular equation is thus found to reduce to

$$\begin{aligned}
(1+\eta^2)\,(1-108\eta+143\eta^2-432\eta^2-270\eta^4-792\eta^5-1{,}023\eta^6\\
-864\eta^7-605\eta^8-900\eta^9+605\eta^{10}-864\eta^{11}\dots-\eta^{18})^2.
\end{aligned}$$

Equating coefficients of $\eta^{10}, \eta^{11}, \eta^{12}, \eta^{13}, \eta^{14}, \eta^{16}, \eta^{18}$, we obtain in turn

$$\begin{array}{llll}
c_5 = 66{,}994, & b_6 = 11{,}556, & c_6 = 187{,}454, & b_7 = 15{,}031,\\
c_7 = 221{,}234, & c_8 = 179{,}655, & c_9 = 88{,}617. &
\end{array}$$

By equating the coefficients of η^{15}, η^{17} we have two equations of verification.

Thus, finally, the modular equation is as written at the beginning of this paragraph, with the numerical coefficients:

$a_1 = 5$, $a_2 = 51$, $a_3 = 133 = 7 . 19$; $b_1 = 1$, $b_2 = -108$, $b_3 = -333 = -3^2 . 37$, $b_4 = 540$, $b_5 = 4{,}806 = 2 . 3^3 . 89$, $b_6 = 11{,}556 = 2^2 . 3^3 . 107$,

$b_7 = 15{,}031$, $c_1 = 3$, $c_2 = 119 = 7 \,.\, 17$, $c_3 = 2{,}073 = 3 \,.\, 691$, $c_4 = 16{,}558 = 2 \,.\, 17 \,.\, 487$, $c_5 = 66{,}994 = 2 \,.\, 19 \,.\, 41 \,.\, 43$, $c_6 = 157{,}454 = 2 \,.\, 11 \,.\, 17 \,.\, 421$, $c_7 = 221{,}234 = 2 \,.\, 13 \,.\, 67 \,.\, 127$, $c_8 = 179{,}655 = 3 \,.\, 5 \,.\, 7 \,.\, 29 \,.\, 59$, $c_9 = 88{,}617 = 3 \,.\, 109 \,.\, 271$.

We can now use complex multiplication for further verification of the coefficients. I have in fact worked out most of the cases, but I reproduce only one. Corresponding to the resolution $37 = 3^2 + 2^2 . 7$, it is easily verified that, if $\tau = 2 + i\sqrt{7}$, $y_5 = y_{35} = x(\tau)$; so that $x(2 + i\sqrt{7}) \equiv \epsilon x(i\sqrt{7})$ is a repeated root of the equation $y = x$. Making this substitution in the modular equation and writing for brevity $z = x^{12}$, we have

$$f(z) \equiv 1 - 2^{18} z^6 - 37Az\,(1 + 2^{12} z^4) - 37Bz^2\,(1 - 2^6 z^2) - (2 + 37C)z^3 = 0,$$

where

$$A = 2\,(a_1 + a_2 + a_3), \quad B = 2\,(b_1 + \ldots + b_6) + b_7, \quad C = 2\,(c_1 + \ldots + c_9).$$

From the modular equation for $n = 7$ we have $x(i\sqrt{7}) = 2^{-1/2}$, whence $z = -2^{-6}$; associated with this by reciprocity we have $z = 1$.

Thus $f(z) \equiv (1 + 63z - 2^6 z^2)^2\,(1 + \lambda z - 2^6 z^2)$, where the second factor corresponds to $z = x^{12}(i\sqrt{37})$.

Giving A its known value 378 and equating coefficients of z, we have $\lambda = -14{,}112$. Equating coefficients of z^2, z^3, we have $B = 47{,}955$, $C = 1{,}465{,}414$, agreeing with preceding results. The residual factor gives

$$8z = -882 + 145\sqrt{37} = (\sqrt{37} - 6)^3;$$

so that

$$2x^4(i\sqrt{37}) = \sqrt{37} - 6,$$

which agrees with Weber's result.

I have to express my thanks to Miss H. P. Hudson, of Newnham College, who has helped me materially by carrying out some of the calculations independently.

KING'S COLLEGE, CAMBRIDGE.

On Translation-Surfaces Connected with a Unicursal Quartic.

By John Eiesland.

In a paper published in Vol. XXIX of the American Journal of Mathematics I have found and discussed all the types of algebraic translation-surfaces that can be generated in four different ways. Surfaces that admit of such fourfold generation were discovered by S. Lie, who in a series of papers* made known their general properties and method of analytical representation. A historical introduction to this interesting subject may be found in a paper published by Georg Scheffers in *Acta Mathematica,* Vol. 28, 1903, where also an independent treatment of certain parts of the theory is given.

With the exception of two theses by R. Kummer and Georg Wiegner† no detailed study of these surfaces has been undertaken, although, as G. Scheffers remarks,‡ such investigations promise sufficient results to justify the effort.

Owing to the large number of types of translation-surfaces admitting of fourfold generation, I limited myself in my former paper to the consideration of algebraic surfaces, reserving the investigation of transcendental surfaces to these and future investigations.

As is well known, all surfaces of this kind are closely connected with a quartic curve, irreducible or not, in the plane at infinity. All surfaces corresponding to projectively equivalent quartics are said to belong to the same *type.* It was found that *all algebraic surfaces correspond to a unicursal quartic having no double or triple points with distinct tangents.* (The correspondence here mentioned will be explained in what follows.)

The remaining unicursal quartics give rise to transcendental surfaces, the study of which is the object of the present paper.

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* *Berichte der Königlich. Säch. Gesells. der Wiss.*, 1896 and 1897. (See Bibliography.)

† Georg Wiegner, Dissertation. Leipzig, 1893.

‡ *Acta Math.*, vol. 28, 1904, p. 90.

Since the whole theory, according to Lie, is intimately bound up with Abel's theorem, the following pages may also be looked upon as a study of Abelian Integrals of the first kind with respect to a unicursal quartic.

The method of constructing translation-surfaces with a fourfold mode of generation is based on a theorem by Lie,* viz.:

If on a translation-surface that can be generated in more than two ways we draw tangents at any point along the four generating curves, the intersection of these tangents with the plane at infinity is a curve of the fourth order.

Conversely, if we suppose given in the plane at infinity a curve of the fourth order, there exist always infinitely many (∞^4) *surfaces generated in four ways, whose tangents along the generating curves cut the plane at infinity along the given curve.*

The coordinates of these surfaces are expressible as the sum of any two Abelian integrals with respect to the four points of intersection of a variable straight line with this quartic curve.

Every direction in space is determined by a point in the plane at infinity; the direction of a line joining a point to a consecutive point is determined whenever the ratios $\frac{dx}{dz}$ and $\frac{dy}{dz}$ are given. We may therefore, with Lie, consider these ratios as coordinates ξ, η in the plane at infinity.

Let there be given in this plane a quartic curve $F(\xi, \eta) = 0$; in order to determine the translation-surface, according to Lie's theorem, we form the Abelian integrals

$$\Phi = \int \frac{\xi d\xi}{F'_{(\eta)}}, \quad \Psi = \int \frac{\eta d\xi}{F'_{(\eta)}}, \quad \mathrm{X} = \int \frac{d\xi}{F'_{(\eta)}},$$

whose limits we fix as follows: We suppose the quartic cut by a fixed and a variable straight line; denoting the abscissas of the point of intersection by $\xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0$ and $\xi_1, \xi_2, \xi_3, \xi_4$ respectively, we choose the former as the lower and the latter as the upper limits, so that we have

$$\Phi_i = \int_{\xi_i^0}^{\xi_i} \frac{\xi_i d\xi_i}{F'_{(\eta_i)}}, \quad \Psi_i = \int_{\xi_i^0}^{\xi_i} \frac{\eta_i d\xi_i}{F'_{(\eta_i)}}, \quad \mathrm{X}_i = \int_{\xi_i^0}^{\xi_i} \frac{d\xi_i}{F'_{(\eta_i)}}.$$

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* *Berichte der Säch. Gesells. der Wiss.*, vol. 48, 1896, p. 197.

Now by Abel's theorem we have (the constants ξ_i^0 being properly chosen):

$$\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \equiv 0,$$
$$\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 \equiv 0,$$
$$X_1 + X_2 + X_3 + X_4 = 0,$$

from which it follows that

$$\Phi_1 + \Phi_2 \equiv -\Phi_3 - \Phi_4,$$
$$\Psi_1 + \Psi_2 \equiv -\Psi_3 - \Psi_4,$$
$$X_1 + X_2 \equiv -X_3 - X_4,$$

so that the equations

$$x = \Phi_1 + \Phi_2, \qquad y = \Psi_1 + \Psi_2, \qquad z = X_1 + X_2$$

represent the same surface as

$$x = -\Phi_3 - \Phi_4, \qquad y = -\Psi_3 - \Psi_4, \qquad z = -X_3 - X_4,$$

a translation-surface generated in four ways, as is seen from the double mode of representation.

If the quartic is irreducible, the integrals Φ_i have the same form; the same is true of the Ψ's and X's. The curves ξ_1 and ξ_2 cover the surface twice and are all parallel to each other and similarly placed. The curve $\xi_1 = \xi_2$ is a special asymptotic line on the surface and the envelope of the curves $\xi_1 = \text{const.}$, $\xi_2 = \text{const.}$ The same is true of the curves ξ_3 and ξ_4 which have for envelope the special asymptotic line $\xi_3 = \xi_4$. The surface may also be considered as the locus of the middle points of all chords of the curve $\xi_1 = \xi_2$ or of the curve $\xi_3 = \xi_4$. It should be noticed that the surface is symmetric with respect to a certain point which, by properly fixing the lower limits of the integrals, may be taken as the origin; *it has therefore a center.*

I.

We shall begin with a quartic having three non-consecutive double points; by a projective transformation (real or imaginary) the curve may be thrown into the form, using x, y instead of ξ, η,

$$x^2 + y^2 - 2axy + x^2y^2 - 2bx^2y - 2cxy^2 = 0, \tag{1}$$

in which the double points are placed at the vertices of the triangle of reference. In order to find a suitable parametric representation we intersect the curve by the hyperbola

$$xy + \rho x + \sigma y = 0, \tag{2}$$

which passes through the double points; let it also pass through the point of intersection of $y = mx$ with the curve, m being one of the roots of the equation $m^2 - 2am + 1 = 0$. This point is easily found to be $\frac{2(b + mc)}{m}$, $2(b + mc)$. Now in order that the hyperbola (2) shall pass through this point, the following relation between ρ and σ must exist:

$$m\sigma + \rho = -2(b + mc).$$

Substituting the value of x from (2) in (1) we have

$$(\sigma^2 + 2c\sigma + 1)y^2 + 2(\rho + a\sigma - b\sigma^2 + c\sigma\rho)y + \sigma^2 + \rho^2 + 2a\sigma\rho = 0,$$

of which $y + m\sigma + \rho$ is a factor. There remains therefore, after dividing the expression,

$$(\sigma^2 + 2c\sigma + 1)y + \rho + (2a - m)\sigma,$$

which gives us the required parametric representation:

$$y = \frac{-\left(\frac{\sigma}{m} + \rho\right)}{\sigma^2 + 2c\sigma + 1} = \frac{(1 - m^2)\rho + 2(b + mc)}{\rho^2 + (4b + 2mc)\rho + m^2 + 4b^2 + 4bmc},$$

$$x = \frac{-(\sigma + m\rho)}{\rho^2 + 2b\rho + 1} = \frac{(1 - m^2)\rho + 2(b + mc)}{m(\rho^2 + 2b\rho + 1)}.$$

We also find

$$dx = \frac{(1 - m^2)\rho^2 + 4(b + mc)\rho + 4b(b + mc) + m^2 - 1}{(\rho^2 + 2b\rho + 1)^2}\,d\rho$$

and

$$F'_{(y)} = \frac{2\sigma y}{(y + \rho)^2}\left[(b\sigma - a)y + \sigma + a\rho\right]$$

$$= \frac{\sigma^2 y}{(y + \rho)^2}\left[\frac{(1 - m^2)\rho^2 + 4(b + mc)\rho + 4b(b + mc) + m^2 - 1}{\sigma^2 + 2c\sigma + 1}\right],$$

$$\frac{dx}{F'_{(y)}} = \frac{-(y + \rho)^2(\sigma^2 + 2c\sigma + 1)}{\sigma^2 y(\rho^2 + 2b\rho + 1)} = \frac{(y + \rho)^2\left(\frac{\sigma}{m} + \rho\right)}{\sigma^2 y^2(\rho^2 + 2b\rho + 1)^2};$$

but

$$x^2 = \frac{\sigma^2 y^2}{(y + \rho)^2} = \frac{(\sigma + m\rho)^2}{(\rho^2 + 2b\rho + 1)^2},$$

hence

$$\left.\begin{aligned}
\frac{dx}{F'_{(y)}} &= -\frac{d\rho}{(1 - m^2)\rho + 2(b + mc)},\\
\frac{x\,dx}{F'_{(y)}} &= -\frac{d\rho}{m(\rho^2 + 2b\rho + 1)},\\
\frac{y\,dx}{F'_{(y)}} &= -\frac{d\rho}{\rho^2 + (4b + 2mc)\rho + m^2 + 4b^2 + 4bmc}.
\end{aligned}\right\} \quad (3)$$

The surface may now be written, putting

$$-mX = X', \qquad -Y = Y', \qquad (m^2-1)\,Z = Z'$$

and dropping the primes,

$$\begin{aligned} X &= \int \frac{d\rho_1}{\rho_1^2 + 2b\rho_1 + 1} + \int \frac{d\rho_2}{\rho_2^2 + 2b\rho_2 + 1}, \\ Y &= \int \frac{d\rho_1}{\rho_1^2 + (4b + 2mc)\,\rho_1 + m^2 + 4b^2 + 4bmc} \\ &\qquad + \int \frac{d\rho_2}{\rho_2^2 + (4b + 2mc)\,\rho_2 + m^2 + 4b^2 + 4bmc}, \\ Z &= \int \frac{d\rho_1}{\rho_1 + \dfrac{2\,(b + mc)}{1 - m^2}} + \int \frac{d\rho_2}{\rho_2 + \dfrac{2\,(b + mc)}{1 - m^2}}. \end{aligned} \tag{3'}$$

The discriminants of the three quadratic equations

$$\begin{aligned} &\rho_1^2 + 2b\rho_1 + 1 = 0, \\ &\rho_1^2 + (4b + 2mc)\,\rho_1 + m^2 + 4b^2 + 4bmc = 0, \\ &m^2 - 2am + 1 = 0 \end{aligned} \tag{4}$$

are $b^2 - 1$, $m^2(c^2 - 1)$, $a^2 - 1$, respectively. If therefore b, c and a be greater than unity, the integration will give rise to logarithmic functions. In case either b, c (or both) is less than unity, while a is greater than unity, antitrigonometric functions instead of logarithmic will be introduced in the equation of the surface. These cases will be discussed later. If a is less than unity the surface (3) is apparently imaginary, although in reality it is as real as in the three preceding cases; this case, therefore, needs separate treatment. For the present we need not distinguish between the different cases; we shall integrate without regard to the sign of the discriminants of (4), it being understood that whenever b, c, or both, are less than unity the surface may be thrown into a real form by introducing trigonometric functions in the coordinates X and Y. This remark is also applicable to the case where all three parameters are less than unity, but, as we have said before, a separate treatment is needed. The geometric interpretation of each of the four cases will also be explained hereafter.

Calling the roots of the first two equations (4) α_1, β_1; α_2, β_2 respectively, we have after integrating:

$$X = \frac{1}{2\sqrt{b^2-1}} \log \frac{(\rho_1-\alpha_1)(\rho_2-\alpha_1)}{(\rho_1-\beta_1)(\rho_2-\beta_1)},$$

$$Y = \frac{1}{2\sqrt{c^2-1}} \log \frac{(\rho_1-\alpha_2)(\rho_2-\alpha_2)}{(\rho_1-\beta_2)(\rho_2-\beta_2)},$$

$$Z = \log(\rho_1-k_1)(\rho_2-k_1), \quad k_1 = -\frac{2(b+mc)}{1-m^2}.$$

By using the transformation $2\sqrt{b^2-1}\,X = X'$, $2\sqrt{c^2-1}\,Y = Y'$, $Z = Z'$, these equations may be written:

$$e^X = \frac{(\rho_1-\alpha_1)(\rho_2-\alpha_2)}{(\rho_1-\beta_1)(\rho_2-\beta_1)}, \quad e^Y = \frac{(\rho_1-\alpha_2)(\rho_2-\alpha_2)}{(\rho_1-\beta_2)(\rho_2-\beta_2)}, \quad e^Z = (\rho_1-k_1)(\rho_2-k_1),$$

or,

$$\begin{aligned} \rho_1\rho_2 - k_1(\rho_1+\rho_2) + k_1^2 - e^Z &= 0, \\ (1-e^X)\rho_1\rho_2 - (\alpha_1-\beta_1 e^X)(\rho_1+\rho_2) + \alpha_1^2 - \beta_1^2 e^X &= 0, \\ (1-e^Y)\rho_1\rho_2 - (\alpha_2-\beta_2 e^Y)(\rho_1+\rho_2) + \alpha_2^2 - \beta_2^2 e^Y &= 0, \end{aligned}$$

from which by elimination we obtain

$$\begin{vmatrix} 1 & -k_1 & k_1^2 - e^Z \\ 1-e^X & -(\alpha_1-\beta_1 e^X) & \alpha_1^2 - \beta_1^2 e^X \\ 1-e^Y & -(\alpha_2-\beta_2 e^Y) & \alpha_2^2 - \beta_2^2 e^Y \end{vmatrix} = 0;$$

which expanded may be written

$$A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Y+Z} + He^{X+Y+Z} = 0, \tag{5}$$

where the coefficients have the following values:

$$\begin{aligned} A &= (\alpha_1-\alpha_2)[\alpha_1\alpha_2 - k_1(\alpha_1+\alpha_2) + k_1^2], \\ B &= (\alpha_2-\beta_1)[\alpha_2\beta_1 - k_1(\alpha_2+\beta_1) + k_1^2], \\ C &= (\beta_2-\alpha_1)[\alpha_1\beta_2 - k_1(\beta_2+\alpha_1) + k_1^2], \\ D &= \alpha_2 - \alpha_1, \\ E &= \beta_1 - \alpha_2, \\ F &= (\beta_1-\beta_2)[\beta_1\beta_2 - k_1(\beta_1+\beta_2) + k_1^2], \\ G &= \alpha_1 - \beta_2, \\ H &= \beta_2 - \beta_1. \end{aligned}$$

These coefficients are not independent; in fact, the following identical relation is easily seen to exist between them:

$$EGAF = HDCB, \tag{6}$$

which is of fundamental importance.

We have then the

THEOREM: *To a unicursal quartic having three double points with distinct tangents there corresponds a translation-surface of the form*

$$A + Be^{X} + Ce^{Y} + De^{Z} + Ee^{X+Z} + Fe^{X+Y} + Ge^{Z+Y} - He^{X+Y+Z} = 0, \tag{5}$$

with the following identical relation between the coefficients:

$$EGAF = HDCB. \tag{6}$$

There exist ∞^3 *types of such surfaces corresponding to the* ∞^3 *projectively non-equivalent quartics with three non-consecutive double points.*

Every surface (5) has a center which is found by putting $X = X' - \xi$, $Y = Y' - \eta$, $Z = Z' - \zeta$ in (5). After this transformation the new coefficients A, B', C', ..., H' must satisfy the following conditions:

$$A = -H', \; B' = -G', \; C' = -E', \; D' = -F';$$

we find then the following equalities:

$$\xi + \eta + \zeta = \log\left(\frac{H}{A}\right),$$

$$\eta + \zeta - \xi = \log\left(\frac{G}{B}\right),$$

$$\zeta + \xi - \eta = \log\left(\frac{E}{C}\right),$$

$$\xi + \eta - \zeta = \log\left(\frac{F}{D}\right).$$

Solving the first three equations, we have

$$\xi = \tfrac{1}{2}\log\left(\frac{BH}{AG}\right), \quad \eta = \tfrac{1}{2}\log\left(\frac{CH}{AE}\right), \quad \zeta = \tfrac{1}{2}\log\left(\frac{EG}{BC}\right), \tag{7}$$

which values are found to satisfy the fourth equation, owing to the relation (6). The surface has now the following simple form:

$$A\left(1 - e^{X+Y+Z}\right) + B'\left(e^{Y+Z} - e^{X}\right) + C'\left(e^{X+Z} - e^{Y}\right) + D'\left(e^{X+Y} - e^{Z}\right) = 0, \tag{8}$$

whose center of symmetry is at the origin.

Remark. A translation $X = X' - 2n\pi i$, $Y = Y' - 2n\pi i$, $Z = Z' - 2n\pi i$, where n is any positive or negative integer, leaves the surface invariant, while a translation $X = X' - n\pi i$, $Y = Y' - n\pi i$, $Z = Z' - n\pi i$ transforms it into a real surface whose center of symmetry is at the point $n\pi i$, $n\pi i$, $n\pi i$, viz.:

$$A(1 + e^{X+Y+Z}) + B'(e^{Y+Z} + e^{X}) + C'(e^{X+Z} + e^{Y}) + D'(e^{X+Y} + e^{Z}) = 0. \quad (9)$$

Before proceeding further we shall introduce a few definitions due to Lie:

If we transform a twisted curve in the space (x, y, z) by the transformation

$$x_1 = \lambda x, \quad y_1 = \mu y, \quad z_1 = \nu z, \qquad (10)$$

we obtain a family of ∞^3 curves which evidently remains invariant for the transformation. We say then that *these curves belong to the same species* (Gattung). The same definition may also be extended to surfaces.*

Another fruitful idea due to Lie is the logarithmic transformation:

$$X = \log x, \quad Y = \log y, \quad Z = \log z, \qquad (11)$$

where (x, y, z) is the so-called logarithmic space.†

Consider now all the curves of the same species,

$$x = \lambda \, . \, \phi(t), \quad y = \mu \, . \, \psi(t), \quad z = \nu \, . \, \chi(t);$$

transforming by (11), we have

$$X = \log \phi(t) + \log \lambda, \quad Y = \log \psi(t) + \log \mu, \quad Z = \log \chi(t) + \log \nu,$$

by which we obtain in the space (X, Y, Z) all the curves that are parallel to each other and similarly placed. Hence:

To all the ∞^3 curves in space (X, Y, Z) obtained by the ∞^3 translations of a twisted curve there correspond in the space (x, y, z) all the ∞^3 curves of the same species. This is also evident from the fact that to a translation in (X, Y, Z) corresponds the affinity transformation (10).

Moreover, to the transformation $X = -X$, $Y = -Y$, $Z = -Z$ (the so-called reflexion, *Spieglung*) corresponds the involutary transformation

$$x_1 = \frac{1}{x}, \quad y_1 = \frac{1}{y}, \quad z_1 = \frac{1}{z}. \qquad (12)$$

The general involutary transformation

$$x_1 = \frac{\lambda}{x}, \quad y_1 = \frac{\mu}{y}, \quad z_1 = \frac{\nu}{z} \qquad (13)$$

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* See Lie-Scheffers, *Berühr. Trans.*, pp. 331 and 334.

† *Ibid.*, p. 356.

may be considered as a succession of the transformations (10) and (12), so that we may say:

To the general involutary transformation (13) *in the space* (x, y, z) *there corresponds a reflexion of all the points of* (X, Y, Z) *with respect to the point* $(\log \lambda, \log \mu, \log \nu)$.

If now we apply the logarithmic transformation to the surface (5), we obtain the cubic surface

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \qquad (14)$$

the coefficients of which satisfy the same relation as before, viz.:

$$EGAF = HDCB. \qquad (6)$$

These ∞^3 surfaces remain invariant by the involutary transformation (13). The transformed surface is:

$$Axyz + \lambda Byz + \mu Cxz + \nu Dxy + \lambda\nu Ey + \mu\lambda Fz + \nu\mu Gx + \lambda\mu\nu H = 0,$$

which is evidently of the same form as (14) with the same relation (6) between the coefficients.

From the above it is easily seen that there is one set of values of λ, μ, ν which will leave the surface invariant, viz.:

$$\lambda = \frac{AG}{BH}, \quad \mu = \frac{EA}{CH}, \quad \nu = \frac{AF}{HD}, \qquad (15)$$

as is easily verified, taking into account the identical relation (6). To the translation curves on (5) correspond the double set of twisted cubics on the surface represented by the equations

$$\begin{aligned} x &= \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}, \\ y &= \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}, \\ z &= (\rho_1 - k_1)(\rho_2 - k_1). \end{aligned} \qquad (16)$$

The curves $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$ constitute a family of curves of the same species which cover the surface doubly and may therefore be considered rather as two families, both made up of curves of the same species. By means of the involutary transformation

$$x_1 = \frac{\lambda}{x}, \quad y_1 = \frac{\mu}{y}, \quad z_1 = \frac{\nu}{z}, \qquad (13)$$

where λ, μ, ν have the values given in (15), we obtain the same surface, but in another analytic form, viz.:

$$\begin{aligned} x_1 &= \lambda \frac{(\rho_3-\beta_1)(\rho_4-\beta_1)}{(\rho_3-\alpha_1)(\rho_4-\alpha_1)}, \\ y_1 &= \mu \frac{(\rho_3-\beta_2)(\rho_4-\beta_2)}{(\rho_3-\alpha_2)(\rho_4-\beta_2)}, \\ z_1 &= \frac{\nu}{(\rho_3-k_1)(\rho_4-k_1)}, \end{aligned} \tag{17}$$

on which the curves ρ_3 and ρ_4 are two families of the same species. We thus see that the involutary transformation (13) has transformed the curves $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$, into $\rho_3 = \text{const.}$, $\rho_4 = \text{const.}$, each pair of families belonging to the same species; while any two curves belonging to different pairs are of different species. We may therefore say:

The surface

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0 \tag{18}$$

contains two pairs of families of curves, each pair consisting of curves of the same species. The surface may be generated by performing on any one of these curves ∞^1 *affinity transformations; that is, the surface admits of a fourfold mode of generation.*[*]

This surface is thus seen to be analogous to the surface

$$Ayz + Bzx + Cxy + Lx + My + Nz = 0,$$

which, as S. Lie has shown,[†] has a similar mode of generation; it has four families of curves; viz., the two sets of generators and two families of cubic curves. By the inverse of the logarithmic transformation this surface is transformed into a translation-surface

$$Ae^{Y+Z} + Be^{Z+X} + Ce^{X+Y} + Le^{X} + Me^{Y} + Ne^{Z} = 0,$$

which corresponds to the case where the quartic degenerates into two intersecting conics.[‡] Whenever this happens the curves belonging to either pair (a set of generators and a family of cubics) *are not of the same species*; this is due to the fact that since the surface (X, Y, Z) corresponds to a degenerate quartic (two conics), the functions Φ_1 and Φ_3 (see p. 172) are of identically the same form, and likewise Φ_2 and Φ_4, while Φ_1 and Φ_2 and also Φ_3 and Φ_4 are not; the same is also true of the Ψ's and X's.

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[*] By fourfold mode of generation we mean in this case that the same surface may be represented in two different ways, namely (16) and (17). The phrase *fourfold mode* applies here to the logarithmic space (x, y, z).

[†] G. Scheffers, *Berühr. Trans.*, Vol. I, pp. 350 and 364.

[‡] *Ibid.*

Conversely, let the surface (17) be given. Since we know that it contains two pairs of families of curves, each pair being of the same species, and that either pair by the reflexion (13) is transformed into the other, we conclude that *the surface*

$$A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Y+Z} + He^{X+Y+Z} = 0 \tag{18}$$

is a translation-surface containing two pairs of families of translation-curves, and thus admits of a fourfold generation.

Let the surface (18) be referred to its center of symmetry as origin, writing it as before

$$A\,(1 - e^{X+Y+Z}) + B'\,(e^{Y+Z} - e^X) + C'\,(e^{X+Y} - e^Z) + D'\,(e^{X+Z} - e^Y) = 0. \tag{8}$$

Putting $X = -X$, $Y = -Y$, $Z = -Z$ and subtracting the result from (8), we have

$$\begin{aligned} A\,(e^{-(X+Y+Z)} - e^{X+Y+Z}) &+ B'\,(e^{-(Y+Z)} - e^{Y+Z} + e^{-X} - e^X) \\ &+ C'\,(e^{-(X+Y)} - e^{X+Y} + e^{-Z} - e^Z) + D'\,(e^{-(X+Z)} - e^{X+Z} - e^Y + e^{-Y}) = 0. \end{aligned}$$

If now we employ the transformation $X = i\,X_1$, $Y = i\,Y_1$, $Z = i\,Z_1$, and reduce, this equation takes the form

$$\begin{aligned} A \sin (X_1 + Y_1 + Z_1) + B'\,[\sin (Y_1 + Z_1) - \sin X_1] + C'\,[\sin (X_1 + Y_1) - \sin Z_1] \\ + D'\,[\sin (X_1 + Z_1) - \sin Y_1] = 0, \end{aligned}$$

which again reduces to

$$\begin{aligned} A \sin \tfrac{1}{2}\,(X_1 + Y_1 + Z_1) + B' \sin \tfrac{1}{2}\,(Y_1 + Z_1 - X_1) + C' \sin \tfrac{1}{2}\,(X_1 + Y_1 - Z_1) \\ + D' \sin \tfrac{1}{2}\,(X_1 + Y_1 - Z_1) = 0, \end{aligned}$$

and finally, putting $\tfrac{1}{2} X_1 = X$, $\tfrac{1}{2} Y_1 = Y$, $\tfrac{1}{2} Z_1 = Z$,

$$\begin{aligned} A \sin (X + Y + Z) + B' \sin (Y + Z - X) + C' \sin (X + Y - Z) \\ + D' \sin (X + Z - Y) = 0. \end{aligned} \tag{19}$$

This transformation, it will be noticed, has no effect on the corresponding quartic in the plane at infinity. In the new space $(i\,X, i\,Y, i\,Z)$ the surface appears as a real surface with three real periods, while in the original space it had three imaginary periods. They both belong to the same type, provided a, b and c in the quartic have constant values. They are, moreover, very different in form: the surface (19) is contained in a cube whose side equals π, and the whole of space being divided into such cubes, each one contains an exact reproduction the surface in the original cube. The surface (8) shows no such periodicity, the periods being imaginary. It is thus seen that *leaving the quartic curve in the plane at infinity invariant, we can express the corresponding surface either as a surface having imaginary periods, or as one having real periods.*

We may express the above results in the following

THEOREM: *To a unicursal quartic having three non-consecutive double points with distinct tangents there corresponds a translation-surface of the form*

$$A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Y+Z} + He^{X+Y+Z} = 0,$$

with the following identical relation between the coefficients:

$$EGAB = HDCB.$$

The surface, when transformed to its center of symmetry as origin, takes the form

$$A(1 - e^{X+Y+Z}) + B'(e^{Y+Z} - e^X) + C'(e^{X+Y} - e^Z) + D'(e^{X+Z} - e^Y) = 0,$$

which by means of the transformation

$$X = 2iX_1, \quad Y = 2iY_1, \quad Z = 2iZ_1,$$

may be put into the form

$$A \sin(X_1 + Y_1 + Z_1) + B' \sin(Y_1 + Z_1 - X_1) + C' \sin(X_1 + Y_1 - Z_1) + D' \sin(X_1 + Z_1 - Y_1) = 0.$$

Remark. If in (19) we put $Y + Z - X = X_1$, $X + Y - Z = Y_1$, $X + Z - Y = Z_1$, the equation becomes

$$A \sin(X_1 + Y_1 + Z_1) + B' \sin X_1 + C' \sin Y_1 + D' \sin Z_1 = 0,$$

which for certain purposes may be simpler and more convenient.

II.

In the case where the three double points have imaginary pairs of tangents (the three vertices of the triangle being conjugate points), the parametric representation of the quartic that we have used (p. 173) becomes inconvenient, if we want the surface in a real form; in fact, k_1 becomes imaginary with m, since m is a root of the equation $m^2 - 2am + 1 = 0$, a now being less than unity. To avoid this difficulty we must find a suitable parametric representation.

We write the quartic as before,

$$x^2 + y^2 - 2axy + x^2y^2 - 2bx^2y - 2cxy^2 = 0, \tag{1}$$

or,

$$\frac{1}{x^2} + \frac{1}{y^2} - \frac{2a}{xy} + 1 - \frac{2b}{y} - \frac{2c}{x} = 0, \tag{2}$$

from which it is seen that a parametric representation of (1) may be found by obtaining one for the conic

$$x_1^2 + y_1^2 - 2\,a\,x_1 y_1 - 2\,b\,y_1 - 2\,c\,x_1 + 1 = 0$$

(see Salmon's Higher Plane Curves, p. 244*), obtained by putting $x_1 = \frac{1}{x}$, $y_1 = \frac{1}{y}$, $z_1 = \frac{1}{z}$ in (2). Since a, b and c are all less than unity, this conic (in general an ellipse) lies wholly inside the triangle of reference. Transforming the origin to the center $\left(\frac{c+ab}{1-a^2},\ \frac{b+ac}{1-a^2}\right)$, we have

$$\bar{x}_1^2 + \bar{y}_1^2 - 2\,a\,\bar{x}_1\bar{y}_1 = \frac{a^2+b^2+c^2+2\,a\,b\,c-1}{1-a^2} = \frac{R^2}{1-a^2},$$

from which it appears that the ellipse, and hence the quartic, is real whenever R^2 is positive. In order to express $\bar{x}_1$ and $\bar{y}_1$ in terms of a variable parameter ρ, we pass a line $y = \rho x + \sigma$ through the point $\left(\frac{R}{\sqrt{1-a^2}},\ 0\right)$ and find the second and variable point of intersection, which is

$$\bar{x}_1 = \frac{(\rho_1^2 - 1)\,R}{\sqrt{1-a^2}\,(1 - 2\,a\,\rho + \rho^2)}, \quad \bar{y}_1 = \frac{2\,\rho\,(a\,\rho - 1)\,R}{\sqrt{1-a^2}\,(1 - 2\,a\,\rho + \rho^2)};$$

and hence,

$$x_1 = \bar{x}_1 + h = \frac{(\rho^2 - 1)\,R}{\sqrt{1-a^2}\,(1 - 2\,a\,\rho + \rho^2)} + \frac{c+ab}{1-c^2},$$

$$y_1 = \bar{y}_1 + k = \frac{2\,\rho\,(a\,\rho - 1)\,R}{\sqrt{1-a^2}\,(1 - 2\,a\,\rho + \rho^2)} + \frac{b+ac}{1-a^2},$$

so that we finally have the following values for x and y on the quartic:

$$\left.\begin{aligned} x &= \frac{(1-a^2)(1-2\,a\,\rho+\rho^2)}{[\sqrt{1-a^2}\,R + c + a\,b]\,\rho^2 - 2\,a\,(c+a\,b)\,\rho + c + a\,b - \sqrt{1-a^2}\,R} \\ y &= \frac{(1-a^2)(1-2\,a\,\rho+\rho^2)}{[2\sqrt{1-a^2}\,a\,R + b + ac]\,\rho^2 - [2\,a\,(b+ac) + 2\sqrt{1-a^2}\,R]\,\rho + b + ac} \end{aligned}\right\} \qquad (2)$$

We also have

$$F'_{(y)} = y - 2\,c\,x\,y + y\,x^2 - a\,x - b\,x\,y = x\sqrt{(b^2-1)\,x^2 + 2\,(c+a\,b)\,x + a^2 - 1}$$

and

$$\frac{dx}{d\rho} = \frac{2\,(1-a^2)\sqrt{1-a^2}\,R\,(a\,\rho^2 - 2\,\rho + 1)}{[(\sqrt{1-a^2}\,R + c + a\,b)\,\rho^2 - 2\,a\,(c+a\,b)\,\rho + c + a\,b - \sqrt{1-a^2}\,R]^2}.$$

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* We refer here to the second edition of this work.

By substituting in $F'_{(y)}$ the value of x in terms of ρ, we have

$$F'_{(y)} = \frac{x\sqrt{1-a^2}\,R\,(a\rho^2 - 2\rho + a)}{(\sqrt{1-a^2}\,R + c + ab)\rho^2 - 2a(c+ab)\rho + c + ab - \sqrt{1-a^2}\,R}.$$

The corresponding surface may now be written:

$$\left.\begin{aligned} X &= 2(1-a^2)\int\frac{d\rho_1}{D_1\rho_1} + 2(1-a^2)\int\frac{d\rho_2}{D_1\rho_2^2}, \\ Y &= 2(1-a^2)\int\frac{d\rho_1}{D_2\rho_1} + 2(1-a^2)\int\frac{d\rho_2}{D_2\rho_2}, \\ Z &= \qquad 2\int\frac{d\rho_1}{1-a\rho_1+\rho_1^2} + \qquad 2\int\frac{d\rho_2}{1-a\rho_2+\rho_2^2}, \end{aligned}\right\} \quad (3)$$

where D_1 and D_2 are the respective denominators of x and y in (2). It should be observed that the discriminant of D_1 and D_2, viz.: $(1-a^2)^2(b^2-1)$ and $(1-a^2)^2(c^2-1)$ respectively, are both negative, b and c being less than unity. We have now, after integrating and transforming in a suitable manner to get rid of extraneous factors,

$$\left.\begin{aligned} X &= \tan^{-1}\frac{\rho_1 - \dfrac{a(c+ab)}{\sqrt{1-a^2}\,R + c + ab}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{\sqrt{1-a^2}\,R + c + ab}} + \tan^{-1}\frac{\rho_2 - \dfrac{a(c+ab)}{\sqrt{1-a^2}\,R + c + ab}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{\sqrt{1-a^2}\,R + c + ab}} \\ Y &= \tan^{-1}\frac{\rho_1 - \dfrac{a(b+ac) + \sqrt{1-a^2}\,R}{2\sqrt{1-a^2}\,aR + b + ac}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{2\sqrt{1-a^2}\,aR + b + ac}} + \tan^{-1}\frac{\rho_2 - \dfrac{a(b+ac) + \sqrt{1-a^2}\,R}{2\sqrt{1-a^2}\,aR + b + ac}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{2\sqrt{1-a^2}\,aR + b + ac}} \\ Z &= \tan^{-1}\frac{\rho_1 - a}{\sqrt{1-a^2}} + \tan^{-1}\frac{\rho_2 - a}{\sqrt{1-a^2}}. \end{aligned}\right\} \quad (4)$$

In order to facilitate elimination we write these equations in the form

$$\left.\begin{aligned} X &= \tan^{-1}\frac{\rho_1 - \alpha_1}{k_1} + \tan^{-1}\frac{\rho_2 - \alpha_1}{k_1}, \\ Y &= \tan^{-1}\frac{\rho_1 - \alpha_2}{k_2} + \tan^{-1}\frac{\rho_2 - \alpha_2}{k_2}, \\ Z &= \tan^{-1}\frac{\rho_1 - a}{\sqrt{1-a^2}} + \tan^{-1}\frac{\rho_2 - a}{\sqrt{1-a^2}}, \end{aligned}\right\} \quad (5)$$

which give rise to the following equations, ρ_1 and ρ_2 being eliminated:

$$A\tan X\tan Y\tan Z + B\tan X\tan Y + C\tan X\tan Z + D\tan Y\tan Z \\ + E\tan X + F\tan Y + G\tan Z = 0, \quad (6)$$

in which the constants $A, B, \ldots, G$ have the following values:

$$\begin{aligned}
A &= (\alpha_1 - \alpha_2)\left[\alpha_1\alpha_2 - a(\alpha_1 + \alpha_2) + 2a^2 - 1\right] + \alpha_1 k_2^2 - \alpha_2 k_1^2 - a(k_2^2 - k_1^2),\\
B &= \sqrt{1 - a^2}\,(\alpha_1^2 - \alpha_2^2 + 2a\alpha_2 - 2a\alpha_1 + k_2^2 - k_1^2),\\
C &= k_2(k_1^2 - \alpha_1^2 + 2\alpha_1\alpha_2 - 2a\alpha_2 + 2a^2 - 1),\\
D &= k_1(\alpha_2^2 - k_2^2 + 2a\alpha_1 - 2\alpha_1\alpha_2 + 1 - 2a^2),\\
E &= 2\sqrt{1 - a^2}\,k_2(\alpha_2 - a),\\
F &= 2\sqrt{1 - a^2}\,k_1(a - \alpha_1),\\
G &= 2k_1k_2(\alpha_1 - \alpha_2).
\end{aligned}$$

It remains now to transform the origin to the center of symmetry and to find the coordinates of this center. If we start with equations (6), putting $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$, and express the conditions that the resulting equation shall reduce to the form

$$A' \tan X \tan Y + B' \tan X \tan Z + C' \tan Y \tan Z + D' = 0,$$

we obtain a set of equations involving $\tan\xi$, $\tan\eta$, $\tan\zeta$ which appear somewhat difficult to solve by ordinary methods. To avoid this difficulty we substitute for the trigonometric functions their exponential values, so that we obtain the following equation of the surface:

$$\left.\begin{aligned}
&(-B - C - D + Ai - Ei - Fi - Gi)\,e^{i(X+Y+Z)}\\
&\qquad + (-B - C - D - Ai + Ei + Fi + Gi)\,e^{-i(X+Y+Z)}\\
&+ (C + D - B - Ai - Fi + Gi)\,e^{i(X+Y-Z)}\\
&\qquad + (C + D - B + Ai + Ei + Fi - Gi)\,e^{-i(X+Y-Z)}\\
&+ (B + D - C + Ai - Ei + Fi - Gi)\,e^{i(X+Z-Y)}\\
&\qquad + (B + D - C - Ai + Ei - Fi + Gi)\,e^{-i(X+Z-Y)}\\
&+ (B + C - D - Ai + Ei - Fi - Gi)\,e^{i(Y+Z-X)}\\
&\qquad + (B + C - D + Ai - Ei + Fi + Gi)\,e^{-i(Y+Z-X)} = 0,
\end{aligned}\right\}\quad(6')$$

which may be written in the form

$$\begin{aligned}A_1 + B_1e^{2iX} + C_1e^{2iY} + D_1e^{2iZ} + E_1e^{2i(X+Z)} + F_1e^{2i(X+Y)} + G_1e^{2i(Y+Z)}&\\ + H_1e^{2i(X+Y+Z)} = 0.&\qquad(6'')\end{aligned}$$

Putting now in (6′) $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$ and expressing the condition of symmetry, viz.:

$$A_1 = H_1,\quad B_1 = G_1,\quad C_1 = E_1,\quad D_1 = F_1,$$

we obtain the following equations:

$$\begin{aligned}
\xi+\eta+\zeta &= \tan^{-1}\frac{B+C+D}{G+F+E-A}, \\
\xi+\eta-\zeta &= \tan^{-1}\frac{C+D-B}{G-A-E-F}, \\
\xi+\zeta-\eta &= \tan^{-1}\frac{B+D-C}{E+G-A-F}, \\
\eta+\zeta-\xi &= \tan^{-1}\frac{B+C-D}{A+G-E-F}.
\end{aligned} \tag{7}$$

Solving these equations, we have, using all four equations (7),

$$\begin{aligned}
\xi &= \frac{1}{2}\left[\tan^{-1}\frac{C+D-B}{G-A-E-F}+\tan^{-1}\frac{B+D-C}{E+G-A-F}\right], \\
\eta &= \frac{1}{2}\left[\tan^{-1}\frac{C+D-B}{G-A-E-F}+\tan^{-1}\frac{B+C-D}{A-E-F+G}\right], \\
\zeta &= \frac{1}{2}\left[\tan^{-1}\frac{B+D-C}{E+G-A-F}+\tan^{-1}\frac{B+C-D}{A-E-F+G}\right].
\end{aligned} \tag{8}$$

Since, moreover, the relation

$$\frac{E_1 G_1}{C_1 B_1}=\frac{H_1 D_1}{A_1 F_1} \tag{9}$$

must necessarily be satisfied, if the surface is to be symmetrical, the equations (7) are all satisfied, so that (9) may be replaced by the equivalent one,

$$\begin{aligned}
\tan^{-1}\frac{B+C+D}{E+F+G-A}=\tan^{-1}\frac{C+D-B}{G-A-E-F}+\tan^{-1}\frac{B+D-C}{E-F+G-H} \\
+\tan^{-1}\frac{B+C-D}{A-E-F+G}.
\end{aligned}$$

We shall not verify this relation, as it would involve long and tedious algebraic calculations; it is moreover unnecessary, its truth being known *a priori*.

The surface (6″) now takes the form

$$\begin{aligned}
A_1(1+e^{2i(X+Y+Z)})+B_1(e^{2i(Y+Z)}+e^{2iX})+C_1(e^{2i(X+Z)}+e^{2iY}) \\
+D_1(e^{2i(Y+Z)}+e^{2iX})=0,
\end{aligned} \tag{10}$$

which may easily be reduced back to the form

$$A'\tan X\tan Y+B'\tan X\tan Z+C'\tan Y\tan Z+D'=0, \tag{11}$$

where the coefficients A', ..., D' are found from the equations

$$\begin{aligned} A'+B'+C'+D' &= \sqrt{(B+C+D)^2+(E+F+G-A)^2}, \\ A'-B'-C'+D' &= \sqrt{(C+D-B)^2+(G-F-E-A)^2}, \\ -A'+B'-C'+D' &= \sqrt{(B+D-C)^2+(A+F-E-G)^2}, \\ -A'-B'+C'+D' &= \sqrt{(B+C-D)^2+(E-F-G-A)^2}. \end{aligned}$$

We have not carried out these calculations in detail, as they do not present any serious difficulties. We have then the

THEOREM: *To a unicursal quartic with three conjugate points there corresponds a translation-surface of the form*

$$A' \tan X \tan Y + B' \tan X \tan Z + C' \tan Y \tan Z + D' = 0. \qquad (11)$$

If we transform (10) by means of the transformation $X' = 2iX$, $Y' = 2iY$, $Z' = 2iZ$, it takes the same form as was obtained in the case where the double points of the quartic have real and distinct tangents [see p. 177, (9)], viz.:

$$A_1(1+e^{X'+Y'+Z'})+B_1(e^{Y+Z}+e^{X})+C_1(e^{X+Z}+e^{Y})+D_1(e^{X+Y}+e^{Z})=0. \qquad (9)$$

Now since a transformation of the form

$$X' = 2iX, \qquad Y' = 2iY, \qquad Z' = 2iZ$$

leaves the quartic in the plane at infinity unaltered, we may collect the result obtained in the following form:

THEOREM: *To a unicursal quartic having non-consecutive double points with distinct tangents, these tangents being either both real, or both imaginary, in pairs, there corresponds a translation-surface which may be thrown into either of the following forms:*

$$A(1+e^{X+Y+Z})+B(e^{Y+Z}+e^{X})+C(e^{X+Z}+e^{Y})+D(e^{X+Y}+e^{Z})=0, \qquad (10)$$

$$A' \tan X \tan Y + B' \tan X \tan Z + C' \tan Y \tan Z + D' = 0. \qquad (11)$$

If we put $X = X' + \pi i$, $Y = Y' + \pi i$, $Z = Z' + \pi i$ in (10) and $X = X' + \pi$, $Y = Y' + \pi$, $Z = Z' + \pi$ in (11), these equations may also be written:

$$A(1-e^{X'+Y'+Z'})+B(e^{Y'+Z'}-e^{X'})+C(e^{X'+Z'}-e^{Y'})+D(e^{X'+Y'}-e^{Z'})=0, \qquad (10')$$

$$A' \tan Z' + B' \tan Y' + C' \tan X' + D' \tan X' \tan Y' \tan Z' = 0; \qquad (11')$$

which are sometimes more convenient, inasmuch as *the center of symmetry is here situated on the surface.*

III.

Quartics Having Two Double Points with Real Tangents and One Conjugate Point.

Let the conjugate point be at $x = 0$, $y = \infty$. We have now to integrate equations (3′), p. 174, on the hypothesis, $a < 1$, $b > 1$, $c > 1$, and after a suitable *real* transformation, in order to avoid extraneous factors, we have

$$\begin{aligned} X &= \tan^{-1}\frac{\rho_1 + b}{\sqrt{1 - b^2}} + \tan^{-1}\frac{\rho_2 + b}{\sqrt{1 - b^2}}, \\ 2Y &= \log\frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}, \\ 2Z &= \log(\rho_1 - k_1)(\rho_2 - k_1), \end{aligned}$$

where α_2, β_2, as before, are the roots of the equation

$$\rho^2 + (4b + 2mc)\rho + m^2 + 4b^2 + 4bmc = 0, \text{ and } k_1 = -\frac{2(b + mc)}{1 - m^2}.$$

Eliminating ρ_1 and ρ_2 we have the equation

$$\begin{vmatrix} 1 & k_1 & k_1^2 - e^{2Z} \\ 1 - e^{2Y} & \alpha_2 - \beta_2 e^{2Y} & \alpha_2^2 - \beta_2^2 e^{2Y} \\ -\tan X & b\tan X + \sqrt{1 - b^2} & (1 - 2b^2)\tan X - 2b\sqrt{1 - b^2} \end{vmatrix} = 0,$$

or, developed,

$$\tan X = \frac{A e^{2(Y+Z)} + B e^{2Y} + C e^{2Z} + D}{A' e^{2(Y+Z)} + B' e^{2Y} + C' e^{2Z} + D'}, \tag{1}$$

where

$$\begin{aligned} A &= -\sqrt{1-b^2}, & A' &= b + \beta_2, \\ B &= \sqrt{1-b^2}(k_1-\beta_2)(k_1+\beta_2+2b), & B' &= (k_1-\beta_2)(1-2b^2-k_1\beta_2-k_1 b-\beta_2 b), \\ C &= \sqrt{1-b^2}, & C' &= -b-\alpha_2, \\ D &= \sqrt{1-b^2}(\alpha_2-k_1)(\alpha_2+k_1+2b), & D' &= (\alpha_2-k_1)(1-2b^2-\alpha_2 k_1-\alpha_2 b-k_1 b). \end{aligned} \tag{2}$$

The equation (1) may be simplified just as in the former case by transforming to the center of symmetry. Putting $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$, and expressing the condition of symmetry, we have

$$\begin{aligned} (A - A'\tan\xi)\, e^{2(\eta+\zeta)} &= -(D - D'\tan\xi), \\ (B - B'\tan\xi)\, e^{2(\eta-\zeta)} &= -(C - C'\tan\xi), \\ (A' + A\tan\xi)\, e^{2(\eta+\zeta)} &= D' + D\tan\xi, \\ (B' + B\tan\xi)\, e^{2(\eta-\zeta)} &= C' + C\tan\xi. \end{aligned} \tag{3}$$

From these equations we find that $\tan\xi$ must be a common root of the following two quadratic equations:

$$\left.\begin{aligned}&\text{(a)}\ (AD' + DA')\tan^2\xi + 2(A'D' - DA)\tan\xi - (AD' + DA') = 0,\\ &\text{(b)}\ (BC' + CB')\tan^2\xi + 2(B'C' - BC)\tan\xi - (BC' + CB') = 0.\end{aligned}\right\} \tag{4}$$

The condition that these equations shall have a common root is

$$(AD' + DA)(B'C' - BC) = (BC' + B'C)(A'D' - DA), \tag{5}$$

which is seen to be identically satisfied by the values of $A, \ldots, D, A', \ldots, D'$ obtained from (2). Calling the roots of (4) α and $-\frac{1}{\alpha}$, we have, by solving,

$$\xi = \tan^{-1}\alpha, \quad \xi = \tan^{-1}\alpha - \frac{\pi}{2},$$

of which either value may be taken without influencing the form of (1) as to symmetry. Solving (3) we have

$$\eta = \tfrac{1}{2}\log\frac{(D'\tan\xi - D)(C'\tan\xi - C)}{(A - A'\tan\xi)(B - B'\tan\xi)},$$

$$\zeta = \tfrac{1}{2}\log\frac{(D'\tan\xi - D)(B - B'\tan\xi)}{(A - A'\tan\xi)(C'\tan\xi - C)}.$$

The surface now reduces to the form

$$\tan X = \frac{A_1(e^{2(Y+Z)} - 1) + B_1(e^{2Y} - e^{2Z})}{A_1'(e^{2(Y+Z)} + 1) + B_1'(e^{2Y} + e^{2Z})}, \tag{6}$$

in which $A_1 = D - D'\tan\xi$, $B_1 = C - C'\tan\xi$, $A_1' = D' + D\tan\xi$, and $B_1' = C' + C\tan\xi$. We have then the

THEOREM: *To a quartic having two double points with distinct and real tangents and one conjugate point there corresponds a translation-surface of the form*

$$\tan X = \frac{Ae^{2(Y+Z)} + Be^{2Y} + Ce^{2Z} + D}{A'e^{2(Y+Z)} + B'e^{2Y} + C'e^{2Z} + D'}, \tag{7}$$

with the following identical relation between the coefficients:

$$(AD' + DA')(B'C' - BC) = (BC' + CB')(A'D' - DA). \tag{5}$$

The surface (6) has two imaginary and one real period. By using the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$, which does not affect the quartic curve, we may transform it into a surface having two real and one imaginary period. We have

$$\frac{e^{-X} - e^{X}}{e^{-X} + e^{X}} = \frac{iA_1(e^{2i(Y+Z)} - 1) + iB_1(e^{2iY} - e^{2iZ})}{A_1'(e^{2i(Y+Z)} + 1) + B_1'(e^{2iY} + e^{2iZ})},$$

which may be written

$$\begin{aligned}&e^{-X}\left[(A_1'-iA_1)e^{2i(Y+Z)}+(B_1'-iB_1)e^{2iY}+(B_1'+iB_1)e^{2iZ}+A_1'+iA_1\right]\\&-e^{X}\left[(A_1'+iA_1)e^{2i(Y+Z)}+(B_1'+iB_1)e^{2iY}+(B_1'-iB_1)e^{2iZ}+A_1'-iA_1\right]=0.\end{aligned}\tag{8}$$

By principle of symmetry this equation may also be written, putting $X=-X$, $Y=-Y$, $Z=-Z$,

$$\begin{aligned}&e^{X}\left[(A_1'-iA_1)e^{-2i(Y+Z)}+(B_1'-iB_1)e^{-2iY}+(B_1'+iB_1)e^{-2iZ}+A_1'+iA\right]\\&-e^{-X}\left[(A_1'+iA_1)e^{-2i(Y+Z)}+(B_1'+iB_1)e^{-2iY}+(B_1'-iB_1)e^{-2iZ}+A_1'-iA\right]=0.\end{aligned}\tag{9}$$

Adding (8) and (9) and introducing the trigonometric equivalents, we have

$$e^{2X}=-\frac{L\tan Y\tan Z+M\tan Y+N\tan Z+P}{L\tan Y\tan Z-M\tan Y-N\tan Z+P},\tag{10}$$

where $L=\frac{B_1'-A_1'}{2}$, $P=\frac{B_1'+A_1'}{2}$, $M=\frac{A_1+B_1}{2}$, $N=\frac{A_1-B_1}{2}$.

IV.

Quartics Having One Double Point with Distinct Tangents and Two Conjugate Points.

We have in this case $b<1$, $c<1$, $a>1$, the conjugate points being $x=0$, $y=\infty$; $x=\infty$, $y=0$. On this hypothesis, integrating equations (3′), p. 174, we have

$$\begin{aligned}X&=\tan^{-1}\frac{\rho_1+b}{\sqrt{1-b^2}}+\tan^{-1}\frac{\rho_2+b}{\sqrt{1-b^2}},\\Y&=\tan^{-1}\frac{\rho_1+2b+mc}{m\sqrt{1-c^2}}+\tan^{-1}\frac{\rho_2+2b+mc}{m\sqrt{1-c^2}},\\Z&=\log(\rho_1-k_1)(\rho_2-k_1).\end{aligned}\tag{1}$$

Eliminating we have

$$e^{2Z}=\frac{A\tan X\tan Y+B\tan X+C\tan Y+D}{A'\tan X\tan Y+B'\tan X+C'\tan Y},\tag{1'}$$

where $A, \ldots, D, A', \ldots, C'$ have the following values:

$$\begin{aligned}A&=(2b+mc)\left[1+k_1^2+bmc+2bk_1+k_1mc\right]+k_1(1-2b^2)\\&\qquad+k_1^2b+m^2(1-c^2)(b_1+k_1),\\B&=m\sqrt{1-c^2}\left[1+k_1^2+2b^2+2bmc+4k_1b+2k_1mc\right],\\C&=\sqrt{1-b^2}\,(2m^2c^2-m^2+2bmc-k_1^2-2k_1b),\\D&=2m\sqrt{1-b^2}\sqrt{1-c^2}\,(b+mc),\\A'&=b+mc,\quad B'=m\sqrt{1-c^2},\quad C'=-\sqrt{1-b^2}.\end{aligned}\tag{2}$$

Transforming the center of symmetry, we have

$$e^{2Z} = \frac{E \tan X \tan Y + F \tan X + G \tan Y + H}{E \tan X \tan Y - F \tan X - G \tan Y + H}, \tag{3}$$

where the coefficients have the following values:

$$\begin{aligned} E &= A - B \tan \eta - C \tan \xi + D \tan \xi \tan \eta = e^{\zeta} (A' - B' \tan \eta - C' \tan \xi), \\ F &= A \tan \eta + B - C \tan \xi \tan \eta - D \tan \xi = -e^{\zeta} (A' \tan \eta + B' - C' \tan \xi \tan \eta), \\ G &= A \tan \xi - B \tan \eta \tan \xi + C - D \tan \eta = -e^{\zeta} (A' \tan \xi - B' \tan \xi \tan \eta + C'), \\ H &= A \tan \xi \tan \eta + B \tan \xi + C \tan \eta + D = e^{\zeta} (A' \tan \xi \tan \eta + B' \tan \xi + C' \tan \eta). \end{aligned} \tag{4}$$

From these equations we obtain by elimination of e^{ζ}

$$\begin{aligned} [(A - D)(B' + C') + A'(B + C)] \tan^2 (\eta + \xi) + 2A'(A - D) \tan (\eta + \xi) \\ - (A - D)(B' + C') - A'(B + C) = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} [(A + D)(C' - B') + A'(C - D)] \tan^2 (\eta - \xi) + 2A'(A + D) \tan (\eta - \xi) \\ - (A + D)(B' - C') - A'(B - C) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} 2A'D \tan (\eta + \xi) \tan (\eta - \xi) + [(B' + C')(A + D) - (B + C) A'] \tan (\eta - \xi) \\ + [A'(B - C) - (B' - C')(A - D)] \tan (\eta + \xi) + 2(BC' - CB') = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} 2(CB' - BC') \tan (\eta + \xi) \tan (\eta - \xi) + [(A - D)(C' - B') - (C - B)A'] \tan (\eta - \xi) \\ + [(B' + C')(A + D) - (B + C) A'] \tan (\eta + \xi) - 2A'D = 0. \end{aligned} \tag{8}$$

From (7) and (8) we easily find

$$\eta = \tfrac{1}{2} \tan^{-1} \frac{CB' - BC' + A'D}{C'A + BD' - A'C},$$

$$\xi = \tfrac{1}{2} \tan^{-1} \frac{BC' + A'D - CB'}{B'A + C'D - BA'}.$$

Calling one of the two reciprocal roots of (5) α, we have

$$\zeta = \log \frac{A - D - (B + C)\alpha}{A' - (B' + C')\alpha},$$

which three coordinates will satisfy all four equations provided the following relation exists:

$$\frac{TU + RS}{(A - D)(B' + C') + A'(B + C)} = \frac{S^2 + U^2}{2A'(D - A)},$$

where

$$R = 2A'D, \quad S = (B' + C')(A + D) - A'(B + C),$$

$$T = A'(B + C) - (B' - C')(A - D), \quad U = 2(BC' - B'C).$$

If we substitute the values $A, \ldots, D, A', \ldots, C'$ from (2) in this relation it is seen to be satisfied identically.*

As in the former case, we may now prove that by means of the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$ we may put (1') in the form

$$\tan X' = \frac{E'(e^{2(X+Y)} - 1) + F'(e^{2X} - e^{2Y})}{E'_1(e^{2(X+Y)} + 1) + F'_1(e^{2X} + e^{2Y})},$$

so that combining the results of III and IV we have the following

THEOREM: *To a unicursal quartic with two double points having distinct tangents and one conjugate point, or two conjugate points and one double point, there correspond* ∞^3 *types of translation-surfaces that can be generated in four different ways. The general equation of these surfaces may be put into either of the two forms:*

$$\text{(a)} \qquad \tan X = \frac{A_1(e^{2(Y+Z)} - 1) + B_1(e^{2Y} - e^{2Z})}{A'_1(e^{2(Y+Z)} + 1) + B'_1(e^{2Y} + e^{2Z})},$$

$$\text{(b)} \qquad e^{2X} = \frac{E \tan Y \tan Z + F \tan Y + G \tan Z + H}{E \tan Y \tan Z - F \tan Y - G \tan Z + H}.$$

The form (a) *is transformed into* (b) *by means of the transformation* $X = iX'$, $Y = iY'$, $Z = iZ'$.

V.

Quartics with One Cusp and Two Double Points.

1. Let the double points have real tangents. Putting $a = 1$ in (3'), p. 174, and remembering that b and c are both greater than unity, we have

$$X = \log \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{(\rho_1 - \beta_1)(\rho_2 - \beta_1)},$$

$$Y = \log \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)},$$

$$Z = \rho_1 + \rho_2.$$

Eliminating ρ_1 and ρ_2, we obtain the surface

$$Z = \frac{(\beta_1^2 - \beta_2^2)e^{X+Y} + (\alpha_2^2 - \beta_1^2)e^X + (\beta_2^2 - \alpha_1^2)e^Y + \alpha_1^2 - \alpha_2^2}{(\beta_2 - \beta_1)e^{X+Y} + (\beta_1 - \alpha_2)e^X + (\alpha_1 - \beta_2)e^Y + \alpha_2 - \alpha_1},$$

and transforming to the center ξ, η, ζ, we find

$$Z = \frac{A(e^{X+Y} - 1) + B(e^X - e^Y)}{A'(e^{X+Y} + 1) + B'(e^X + e^Y)}, \qquad (1)$$

---

* The details of the algebraic work have been omitted as unnecessary.

where

$$A = [\alpha_2^2 - \alpha_1^2 + \zeta(\alpha_2 - \alpha_1)], \quad B = [\alpha_1^2 - \beta_2^2 + \zeta(\alpha_1 - \beta_2)]\, e^{\eta},$$
$$A' = \alpha_2 - \alpha_1, \qquad B' = (\alpha_1 - \beta_2)\, e^{\eta},$$

the coordinates of the center of symmetry being

$$\xi = \tfrac{1}{2} \log \frac{(\alpha_2 - \alpha_1)(\alpha_1 - \beta_2)}{(\beta_2 - \beta_1)(\beta_1 - \alpha_2)}, \qquad \eta = \tfrac{1}{2} \log \frac{(\alpha_2 - \alpha_1)(\beta_1 - \alpha_2)}{(\alpha_1 - \beta_2)(\beta_2 - \beta_1)},$$
$$\zeta = -\frac{\alpha_1 + \alpha_2 + \beta_1 + \beta_2}{2}.$$

2. When the double points are conjugate points, that is $b < 1$, $c < 1$, the surface, when transformed to its center of symmetry as origin, takes the form

$$Z = \frac{A \tan X + B \tan Y}{A' \tan X \tan Y + B'}, \tag{2}$$

which may be derived from equations (3′), p. 174, by putting $m = 1$. Hence the

THEOREM: *To a unicursal quartic with one cusp, and two double points whose tangents may be either real or imaginary, there correspond* ∞^2 *types of translation-surfaces that can be generated in four different ways. The equation of these surfaces may be thrown into either of the two following forms* (corresponding to real and imaginary pairs of tangents):

$$Z = \frac{A(e^{X+Y} - 1) + B(e^X - e^Y)}{A'(e^{X+Y} + 1) + B'(e^X + e^Y)}, \tag{1}$$
$$Z = \frac{A \tan X + B \tan Y}{A' \tan X \tan Y + B'}. \tag{2}$$

3. If only one of the double points is a conjugate point, we have, since now $a = 1$, $b < 1$, $c > 1$ (p. 174, (3′)),

$$X = \tan^{-1} \frac{\rho_1 + b}{\sqrt{1 - b^2}} + \tan^{-1} \frac{\rho_2 + b}{\sqrt{1 - b^2}},$$
$$Y = \log \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)},$$
$$Z = \rho_1 + \rho_2,$$

which gives rise to the following equations, eliminating ρ_1 and ρ_2:

$$Z = \frac{(2b^2 - 1 - \beta_2^2) \tan X . e^Y + (\alpha_2^2 + 1 - 2b^2) \tan X + 2b\sqrt{1 - b^2}\, e^Y - 2b\sqrt{1 - b^2}}{-(b + \beta_2) \tan X . e^Y + (b + \alpha_2) \tan X - \sqrt{1 - b^2}\, e^Y + \sqrt{1 - b^2}}$$

which by transformation to the center of symmetry takes the form

$$Z = \frac{A \tan X . (e^Y + 1) + B (e^Y - 1)}{A' \tan X . (e^Y - 1) + B' (e^Y + 1)}, \tag{3}$$

so that we have the

THEOREM: *To a quartic having one cusp, one double point with real tangents and one conjugate point correspond* ∞^2 *translation-surfaces of the form*

$$Z = \frac{A \tan X . (e^Y + 1) + B (e^Y - 1)}{A' \tan X . (e^Y - 1) + B' (e^Y + 1)}. \tag{3}$$

It will be noticed that in this case the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$ leaves the surface in the same form as before.

VI.

Quartics with a Double Point and Two Cusps.

1. The double point has a pair of real tangents. In this case we have $b = c = 1$ and $a > 1$. Equations (3′), p. 174, give us by integrating:

$$\begin{aligned} X &= \frac{1}{\rho_1 + 1} + \frac{1}{\rho_2 + 1}, \\ Y &= \frac{1}{\rho_1 + 2 + m} + \frac{1}{\rho_2 + 2 + m}, \\ Z &= \log \left(\rho_1 + \frac{2}{1 - m}\right) \left(\rho_2 + \frac{2}{1 - m}\right), \end{aligned} \tag{1}$$

from which we obtain the surface

$$[X - Y - (1 + m) XY] e^Z - \frac{(m+1)^2 (2m - 1)}{(1 - m)^2} X + \frac{m (m + 1)^2 (2 - m)}{(1 - m)^2} Y$$
$$+ \frac{m (1 + m)^3}{(1 - m)^2} XY + 2 (1 + m) = 0,$$

which may be written, putting $(1 - m)^2 e^Z = e^{Z'}$,

$$e^{Z'} = \frac{(m+1)^2 (2m - 1) X + m (m+1)^2 (m - 2) e^Y - m (1 + m)^3 XY - 2 (1 + m) (1 - m)^2}{X - Y - (1 + m) XY}.$$

We now put $Z' = Z - \log k_3$, $X = X' + k_1$, $Y = Y' + k_2$, k_3 being a positive quantity, and express the condition that the coefficients of X and Y in the numerator shall equal the coefficients of X and Y in the denominator taken with opposite signs, while the absolute term and the coefficient of XY in the numerator shall equal the corresponding terms in the denominator. We thus obtain the following four equations:

$$\begin{aligned} k_1 k_3 (1+m)^2(2m-1) + k_2 k_3 m (1+m)^2 (m-2) - k_1 k_2 k_3 m (1+m)^3 & \\ - 2k_3(1+m)(1-m)^2 &= k_1 - k_2 - k_1 k_2 (1+m), \\ k_3 m (1+m)^3 &= 1+m, \\ k_3(1+m)^2(2m-1) - k_2 k_3 m (1+m)^3 &= -1 + k_2(1+m), \\ k_3 m (1+m)^2 (m-2) - k_1 k_3 m (1+m)^3 &= 1 + k_1(1+m). \end{aligned}$$

Solving the three last equations, we obtain

$$k_1 = \frac{m-3}{2(1+m)}, \quad k_2 = \frac{3m-1}{2m(1+m)}, \quad k_3 = \frac{1}{m(1+m)^2},$$

which, substituted in the first, reduces it to an identity. The equation of the surface is:

$$e^Z = \frac{2(1-m)^2 - \frac{m-1}{2m} X - \frac{m-1}{2} Y + (1+m) XY}{2(1-m)^2 + \frac{m-1}{2} X + \frac{m-1}{2} Y + (1+m) XY},$$

or, putting $\frac{m-1}{2m} X$ equal to a new X and $\frac{m-1}{2} Y$ equal to a new Y,

$$e^Z = \frac{2(1-m)^2 - X - Y + \frac{4m(1+m)}{(1-m)^2} XY}{2(1-m)^2 + X + Y + \frac{4m(1+m)}{(1-m)^2} XY}.$$

If now we make use of the transformation $X = X' + iY'$, $Y = X' - iY'$, which amounts to transforming the quartic into a limaçon with a conjugate point, we obtain a surface which has a striking resemblance to the Cardioid surface obtained in my previous paper.* The equation of this surface is:

$$e^Z = \frac{2(1-m)^2 - 2X + \frac{4m(1+m)}{(1-m)^2}(X^2+Y^2)}{2(1-m)^2 + 2X + \frac{4m(1+m)}{(1-m)^2}(X^2+Y^2)}. \tag{3}$$

Every section parallel to the XY-plane is a circle which for $Z=0$ becomes the Y-axis. If we put $e^Z = k$, $A = 2(1-m)^2$, $B = \frac{4m(1+m)}{(1-m)^2}$, this section may be written

$$B(k-1)(X^2+Y^2) + 2(k+1)X + A(k-1) = 0,$$

---

* *American Journal of Mathematics*, vol. 29, p. 378. (See plate of model.)

which shows that *if the circular section be real, we must have* $AB < \left(\frac{k+1}{k-1}\right)^2$, *so that to any given type, that is, for any given value of m, the surface will have two umbilical points on the Z-axis at equal distances above and below the origin.* This point will be at infinity when $8m(1+m)=1$, or $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3}{2}}$. *If the product AB is* < 1, *there will be no real umbilical point. If AB is* > 1, *this point is determined by the equation*

$$\left(\frac{e^Z+1}{e^Z-1}\right)^2 = AB = 8m(1+m),$$

which, regarded as an equation determining Z, has two real and finite roots.

Examples: 1. $m = \frac{-1}{2}$. The surface extends to infinity in both directions along the Z-axis and has no umbilical points (Fig. 1).

2. $m = -2$. The surface has two umbilical points at $Z = \pm \log \frac{5}{3}$ (Fig. 2). In both cases the projection of the surface on the XZ-plane has been given.

The surface (3) has an imaginary period, but if we transform to the imaginary space (iX, iY, iZ), we obtain one having a real period. Writing the surface in the form

$$e^Z = \frac{A - 2X + B(X^2+Y^2)}{A + 2X + B(X^2+Y^2)}$$

and using the transformation, we have

$$\frac{e^{2iZ}-1}{e^{2iZ}+1} = \frac{1}{i}\tan Z = \frac{-4Xi}{A - 4B(X^2+Y^2)},$$

or,

$$\tan Z = \frac{4X}{A - 4B(X^2+Y^2)}, \tag{4}$$

one branch of which is contained entirely in a space between the parallel planes $Z = -\pi$ and $Z = \pi$. Any section $\tan Z =$ const. is the circle

$$X^2 + Y^2 + \frac{X}{Bk} = \frac{A}{4B},$$

or,

$$\left(X + \frac{1}{2Bk}\right)^2 + Y^2 = \frac{1}{4B^2}\left(AB + \frac{1}{k^2}\right),$$

from which it is obvious that, whenever AB is positive, the surface will extend to infinity along the Z-axis. If, however, AB is negative, the surface will become imaginary somewhere between $Z=0$ and $Z=\pm\frac{\pi}{2}$, so that there will be an umbilical point at $k=\pm\frac{1}{\sqrt{-AB}}$.

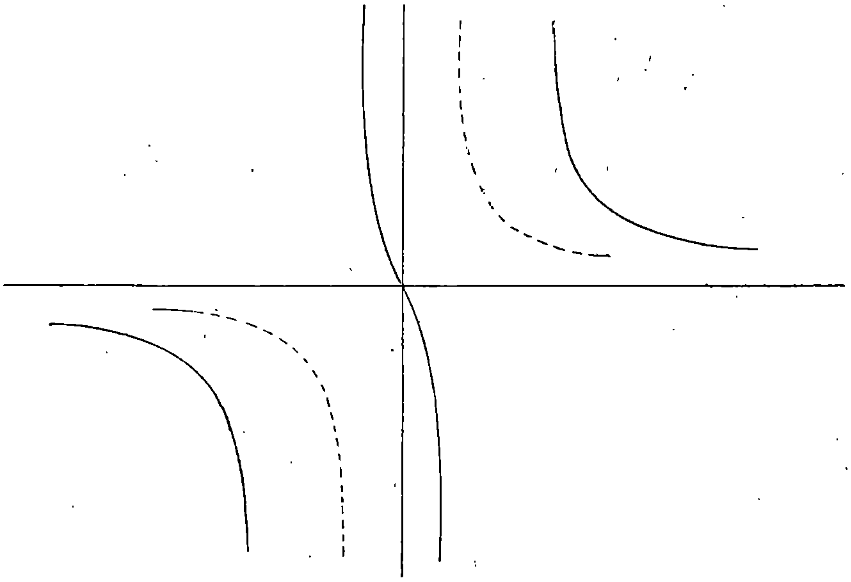

FIG. 1.

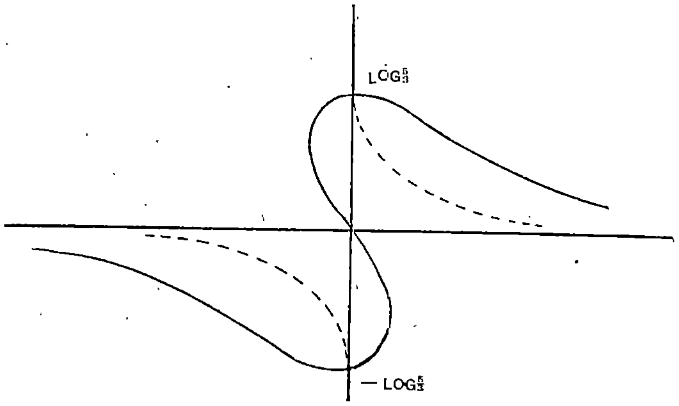

FIG. 2.

Example 1. If $A=4B$, m is equal to 0.07 nearly; for $Z=\frac{\pi}{2}$ we obtain the unit circle $X^2+Y^2=1$ (Fig. 3).

Example 2. Let $AB = -1$, $m = \frac{-1 \pm \sqrt{\frac{1}{2}}}{2}$. The umbilical point is $Z = \pm \frac{\pi}{4}$ (Fig. 4). In both examples the locus of the centers of the circular sections has been indicated by the dotted curve.

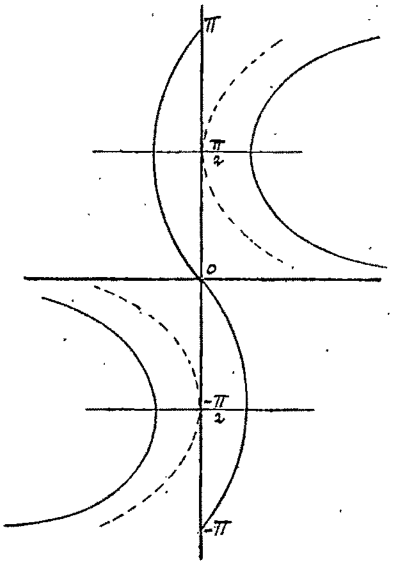

FIG. 3.

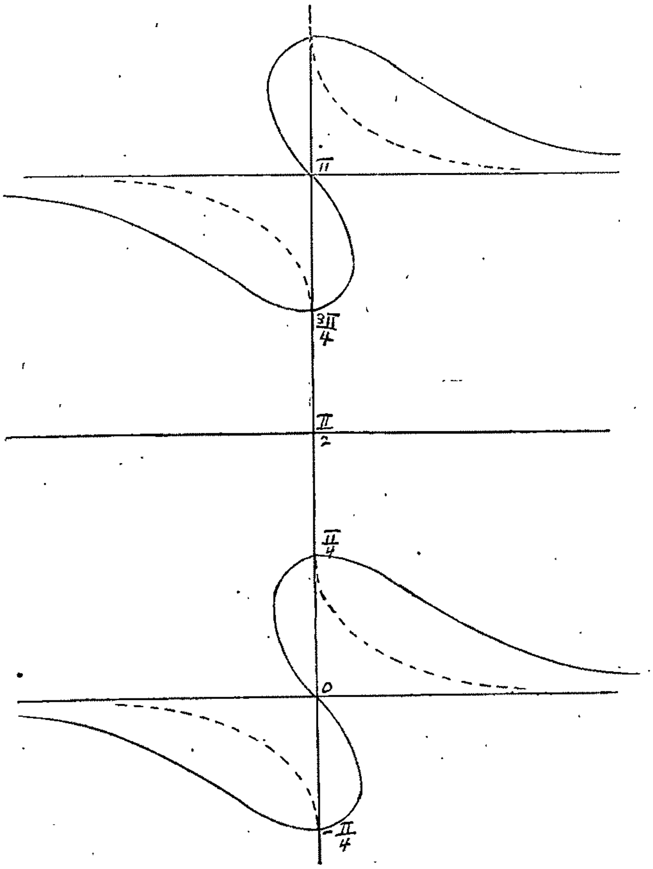

FIG. 4.

VII.

Quartics Having Two Cusps and a Conjugate Point.

In this case we put $m=1$, $a=1$, $c=1$, in (3′), p. 174, while b is less than unity. Integrating we have, omitting extraneous factors,

$$X = \tan^{-1} \frac{\rho_1 + b}{\sqrt{1 - b^2}} + \tan^{-1} \frac{\rho_2 + b}{\sqrt{1 - b^2}},$$

$$Y = \frac{1}{\rho_1 + 1 + 2b} + \frac{1}{\rho_2 + 1 + 2b},$$

$$Z = \rho_1 + \rho_2,$$

from which by elimination we obtain

$$\tan X = \frac{(Z + 2b)\, Y \sqrt{1 - b^2}}{2\,(1 + b)^2\, Y + (b + 1)\, YZ - Z - 2 - 4b};$$

transforming, putting $Z + 2b = Z'$, this may be written:

$$\tan X = \frac{\sqrt{1 - b^2}\, YZ}{2\,(1 + b)\, Y + (1 + b)\, YZ - Z - 2b - 2}.$$

The center of symmetry may now be found just as before. We have

$$\frac{\tan X + k_1}{1 - k_1 \tan X} = \frac{(Y + k_2)\,(Z + k_3)\,(1 - k_1 \tan X)\sqrt{1 - b^2}}{2(1 + b)(Y + k_2) + (1 + b)(Y + k_2)(Z + k_3) - Z - k_3 - 2b - 2},$$

in which the terms in $YZ \tan X$, $\tan X$, Y, Z must vanish. We have therefore the four equations:

$$\begin{aligned}
&-\sqrt{1 - b^2}\, k_1 - (1 + b) = 0,\\
&\sqrt{1 - b^2}\, k_3 - 2k_1\,(1 + b) - k_1 k_3\,(1 + b) = 0,\\
&\sqrt{1 - b^2}\, k_2 - k_1 k_2\,(1 + b) + k_1 = 0,\\
&-\sqrt{1 - b^2}\, k_1 k_2 k_3 + k_3 + 2b + 2 - 2\,(1 + b)\, k_2 - (1 - b)\, k_2 k_3 = 0.
\end{aligned}$$

Solving, we find

$$k_1 = -\frac{1 + b}{\sqrt{1 - b^2}}, \quad k_2 = \tfrac{1}{2}, \quad k_3 = -(1 + b),$$

which values satisfy the fourth equation. The equation now reduces to the form

$$\tan X = \frac{(1 + b)\,(1 + 2b) - 2\,(1 + b)\, YZ}{\sqrt{1 - b^2}\,(Z - 2\,(1 + b)\, Y)},$$

which may be simplified by putting $\dfrac{\sqrt{1 - b^2}}{1 + b} Z = Z'$, $-2\sqrt{1 - b^2}\, Y = Y'$, so that we have

$$\tan X = \frac{(1 + 2b) + \dfrac{1}{1 - b} ZY}{Z + Y}.$$

Every section $X = \text{const.}$ is a rectangular hyperbola; one branch of the surface is contained entirely in the space between the planes $X = \dfrac{\pi}{2}$, $X = -\dfrac{\pi}{2}$. If we transform the surface, using the transformation $Z = Z' + iY'$, $Y = Z' - iY'$, we have

$$\tan X = \frac{1 + 2b + \dfrac{1}{1 - b}(Z^2 + Y^2)}{2Z}, \tag{5}$$

which, by putting $X = X + \frac{\pi}{2}$, is seen to be of the same form as (4), p. 195. It appears then that in this case no new types are obtained. As in all other cases, the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$ will transform (5) into a form involving e^X instead of $\tan X$, viz.:

$$e^X = \frac{A + 2Z + B(Y^2 + Z^2)}{A - 2Z + B(Y^2 + Z^2)}.$$

THEOREM: *To a unicursal quartic with two cusps and one double point with a real or imaginary pair of tangents there correspond* ∞^1 *types of translation-surfaces that can be generated in four different ways. These surfaces may by proper transformations be brought into either of the two forms:*

$$\tan Z = \frac{A + B(X^2 + Y^2)}{2X},$$

$$e^Z = \frac{A' - 2X + B'(X^2 + Y^2)}{A' + 2X + B'(X^2 + Y^2)}.$$

We shall now collect the results obtained in the following table:

	THE PLANE (xy) AT INFINITY.	SPACE (X, Y, Z).
I.	a. Quartic curve with three double points having real tangents.	a. $A(1 - e^{X+Y+Z}) + B(e^{Y+Z} - e^X) + C(e^{X+Z} - e^Y) + D(e^{X+Y} - e^Z) =$
	b. Quartics with three conjugate points.	b. $A \tan X \tan Y + B \tan X \tan Z + C \tan Y \tan Z + D = 0.$
	c. Quartics having three double points of which two are conjugate points.	c. $e^{2X} = \frac{L \tan X \tan Z + M \tan Y + N \tan Z + P}{L \tan X \tan Z - M \tan Y - N \tan Z + P}.$
	d. Quartics with three double points of which one is a conjugate point.	d. $\tan 2X = \frac{A(e^{Y+Z} - 1) + B(e^Y - e^Z)}{A'(e^{Y+Z} + 1) + B'(e^Y + e^Z)}.$
II.	a. Quartics having one cusp and two double points, both having distinct and real tangents.	a. $X = \frac{A(e^{Z+Y} - 1) + B(e^Z - e^Y)}{A'(e^{Z+Y} + 1) + B'(e^Z + e^Y)}.$
	b. Quartics having one cusp and two double points, one of which is a conjugate point.	b. $X = \frac{A \tan Z(e^Y + 1) + B(e^Y - 1)}{A' \tan Z(e^Y - 1) + B'(e^Y + 1)}.$
III.	a. Quartics having two cusps and one double point.	a. $\tan Z = \frac{A + B(X^2 + Y^2)}{2X}.$
	b. Quartics with two cusps and a conjugate point.	b. $e^Z = \frac{A' - 2X + B'(X^2 + Y^2)}{A' + 2X + B'(X^2 + Y^2)}.$

VIII.

Quartics with a Triple Point (Real Tangents).

A quartic with a triple point may be written $yu_3 = u_4$, where u_3 is homogeneous of the third degree in x and z and u_4 of the fourth degree in the same variables. If $y = 0$ be taken as a double tangent and $x = 0$, $z = 0$ be two of the tangents at the triple point, the curve will take the form

$$xyz(x - \alpha z) = (x^2 + k_1 xz + k_2 z^2)^2. \tag{1}$$

Putting now $\alpha z = z'$, $y = \alpha y'$ $x = x'$, this equation reduces to

$$xyz(x - z) = (x^2 + axz + bz^2)^2,$$

where $a = \frac{k_1}{\alpha}$, $b = \frac{k_2}{\alpha^2}$; or, in Cartesian coordinates,

$$xy(x - 1) = (x^2 + ax + b)^2. \tag{2}$$

The corresponding translation-surface may now be written:

$$\begin{aligned} X &= \int \frac{dx_1}{x_1 - 1} + \int \frac{dx_2}{x_2 - 1}, \\ Y &= \int \frac{(x_1^2 + ax_1 + b)^2}{x_1^2(x_1 - 1)^2} + \int \frac{(x_2^2 + ax_2 + b)^2}{x_2^2(x_2 - 1)^2}, \\ Z &= \int \frac{dx_1}{x_1(x_1 - 1)} + \int \frac{dx_2}{x_2(x_2 - 1)}, \end{aligned} \tag{3}$$

from which we derive the following equalities:

$$\left.\begin{aligned} X &= \log(x_1 - 1)(x_2 - 1), \quad Z = \log\frac{(x_1 - 1)(x_2 - 1)}{x_1 x_2}, \\ x_1 + x_2 &= e^{X+Z} - e^X + 1, \qquad x_1 x_2 = e^{X-Z}; \end{aligned}\right\} \tag{4}$$

$$\begin{aligned} Y = x_1 + x_2 + 2(a + 1)X + 2bZ - (a + 1)^2\left(\frac{1}{x_1 - 1} + \frac{1}{x_2 - 1}\right) - 2b(a + 1)Z \\ -2b(a+1)\left(\frac{1}{x_1-1}+\frac{1}{x_2-1}\right)+b^2\left[-\frac{1}{x_1}-\frac{1}{x_2}+2(X-Z)-\left(\frac{1}{x_1-1}+\frac{1}{x_2-1}\right)-2X\right]; \end{aligned}$$

so that we have from (4)

$$\begin{aligned} Y - 2(a + 1)X + 2b(a + b)Z = x_1 + x_2 - (a + b + 1)^2\left[\frac{x_1 + x_2 - 2}{(x_1 - 1)(x_2 - 1)}\right] \\ - b^2\left(\frac{x_1 + x_2}{x_1 x_2}\right). \end{aligned} \tag{5}$$

Substituting from (4) and putting the left-hand side of (5) equal to a new Y, we have

$$Y = 1 + e^{X-Z} - e^{X} - (a+b+1)^2\left[\frac{e^{X-Z} - e^{X} - 1}{e^{X}}\right] - b^2\left[\frac{e^{X-Z} - e^{X} + 1}{e^{X-Z}}\right],$$

which may be reduced to the form

$$Y = e^{X-Z} - b^2 e^{Z-X} - [e^{X} - (a+b+1)^2 e^{-X}] + b^2\left[e^{Z} - \frac{(a+b+1)^2}{b^2} e^{-Z}\right].$$

Putting $X = X' + k_1$, $Y = Y'$, $Z = Z' + k_3$, we shall determine k_1 and k so as to make the surface symmetrical with respect to the origin; the center is easily found to be

$$k_1 = \log(a+b+1), \quad k_2 = 0, \quad k_3 = \frac{a+b+1}{b}, \quad (a+b+1 \neq 0),$$

so that we have, finally,

$$Y = b\,(e^{X-Z} - e^{Z-X}) - (a+b+1)(e^{X} - e^{-X}) + b\,(a+b+1)(e^{Z} - e^{-Z}), \qquad (6)$$

which surface has two imaginary periods.

By means of the well-known transformation $X = iX'$, $Y = iY'$, $Z = iZ'$, it may be transformed into the form

$$Y = b\sin(X-Z) - (a+b+1)\sin X + b\,(a+b+1)\sin Z, \qquad (6')$$

which has two real periods.

1. If $a + b < -1$, the center of symmetry is imaginary.

2. If $a^2 - 4b = 0$, *the quartic has a point of undulation at* $x = -\frac{a}{2}$, $y = 0$; in this case the surface (6) becomes

$$Y = \frac{a^2}{4}(e^{X-Z} - e^{Z-X}) - \left(\frac{a}{2}+1\right)^2(e^{X} - e^{-X}) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2(e^{Z} - e^{-Z}), \qquad (7)$$

or,

$$Y = \frac{a^2}{4}\sin(X-Z) - \left(\frac{a}{2}+1\right)^2\sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2\sin Z. \qquad (7')$$

THEOREM: I. *To a quartic having a triple point with real tangents correspond* ∞^2 *translation-surfaces of the form*

$$Y = b\,(e^{X-Z} - e^{Z-X}) - (a+b+1)(e^{X} - e^{-X}) + b\,(a+b+1)(e^{Z} - e^{-Z}), \qquad (6)$$

or,

$$Y = b\sin(X-Z) - (a+b+1)\sin X + b\,(a+b+1)\sin Z. \qquad (6')$$

II. *To a quartic with a triple point and a point of undulation correspond* ∞^1 *translation-surfaces of the form*

$$Y = \frac{a^2}{4}(e^{X-Z} - e^{Z-X}) - \left(\frac{a}{2}+1\right)^2 (e^X - e^{-X}) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2 (e^Z - e^{-Z}), \tag{7}$$

or,

$$Y = \frac{a^2}{4} \sin (X - Z) - \left(\frac{a}{2}+1\right)^2 \sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2 \sin Z. \tag{7'}$$

IX.

Quartics Having a Triple Point with One Real and Two Imaginary Tangents.

We put the quartic in the form

$$yz\,(x^2 + z^2) = (x^2 + axz + bz^2)^2, \tag{1}$$

or, putting $z = 1$,

$$y\,(x^2 + 1) = (x^2 + ax + b)^2, \tag{2}$$

from which we derive the surface (x_1 and x_2 being the parameters):

$$\begin{aligned} X &= \log (x_1^2 + 1)(x_2^2 + 1), \quad Z = \tan^{-1} x_1 + \tan^{-1} x_2, \\ Y &= x_1 + x_2 + a \log (x_1^2 + 1)(x_2^2 + 1) + 2\,(b - 1)(\tan^{-1} x_1 + \tan^{-1} x_2) \\ &\quad + a^2 (\tan^{-1} x_1 + \tan^{-1} x_2) - a\,(b - 1)\left[\frac{1}{x_1^2 + 1} + \frac{1}{x_2^2 + 1}\right] \\ &\quad + \left[\frac{(b-1)^2 - a^2}{2}\right]\left[\frac{x_1}{x_1^2+1} + \frac{x_2}{x_2^2+1}\right] + \left[\frac{(b-1)^2 - a^2}{2}\right](\tan^{-1} x_1 + \tan^{-1} x_2); \end{aligned} \tag{3}$$

so that we have

$$\begin{aligned} Y - 2aX - [2\,(b - 1)\,Z + a^2 Z + \tfrac{1}{2}((b - 1)^2 - a^2)\,Z] &= \sin Z e^X \\ - a\,(b-1)\,[\sin^2 Z + 2e^{-X} \cos Z] &+ \left[\frac{(b-1)^2 - a^2}{2}\right] \sin Z\,(2e^{-X} - \cos Z), \end{aligned}$$

or,

$$Y = e^X e^Z + \frac{a\,(b-1)}{2}[\cos 2Z - 4e^{-X} \cos Z] + \frac{(b-1)^2 - a^2}{2}[\sin Z\,(2e^{-X} - \cos Z)].$$

Transforming to the center of symmetry, putting $X = X' + k_1$, $Y = Y'$, $Z = Z' + k_3$, we easily find

$$k_1 = -\tfrac{1}{2} \log [a^2 + (b - 1)^2], \quad k_3 = \tan^{-1} \frac{a}{b - 1};$$

and the surface reduces to the form

$$Y = (b-1) \sin Z\,.\,(e^X + e^{-X}) + a \cos Z\,.\,(e^X - e^{-X}) - \tfrac{1}{4}\,[(b-1)^2 + a^2] \sin 2Z.^{*} \tag{4}$$

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[*] Here, as elsewhere, we have omitted the somewhat long algebraic calculations. As a check we have used throughout the property of symmetry which characterises all these surfaces.

If the quartic in addition has a point of undulation, we have $a^2 = 4b$, so that (4) reduces to

$$Y = (b-1)\sin Z(e^X + e^{-X}) + 2\sqrt{b}\cos Z(e^X - e^{-X}) - \tfrac{1}{4}(b+1)^2 \sin 2Z.$$

X.

Quartics with a Triple Point, Two of the Tangents Being Coincident.

The quartic may be written

$$yxz^2 = (x^2 + azx + bz^2)^2,$$

in which a may be reduced to unity,

$$yxz^2 = (x^2 + zx + bz^2)^2,$$

or,

$$yx = (x^2 + x + b)^2.$$

The corresponding surface is

$$Y = \tfrac{1}{3}X^3 - bX(e^Z + e^{-Z}), \tag{1}$$

the center of symmetry being $k_1 = 0$, $k_2 = 0$, $k_3 = \log b$. If $1 - 4b = 0$, the quartic will have a point of undulation, in which case the surface (1) becomes

$$Y = \tfrac{1}{3}X^3 - \tfrac{1}{4}X(e^Z + e^{-Z}).$$

We may now express the results obtained in VIII, IX and X thus:

THEOREM: I. *To a quartic with a triple point, the tangents being all real, there correspond* ∞^2 *types of translation-surfaces of the form*

$$Y = b(e^{X-Z} - e^{Z-X}) - (a+b+1)(e^X - e^{-X}) + b(a+b+1)(e^Z - e^{-Z}), \tag{6}$$

or,

$$Y = b\sin(X-Z) - (a+b+1)\sin X + b(a+b+1)\sin Z.$$

II. *To a quartic with a triple point, having real tangents and also a point of undulation, correspond* ∞^1 *types of translation-surfaces of the form*

$$Y = \frac{a^2}{4}(e^{X-Z} - e^{Z-X}) - \left(\frac{a}{2}+1\right)^2(e^X - e^{-X}) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2(e^Z - e^{-Z}), \tag{7}$$

or,

$$Y = \frac{a^2}{4}\sin(X-Z) - \left(\frac{a}{2}+1\right)^2\sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2\sin Z.$$

III. *To a quartic with a triple point, one pair of whose tangents are imaginary, there correspond* ∞^2 *translation-surfaces of the form*

$$Y = (b-1)\sin Z.(e^X + e^{-X}) + a\cos Z.(e^X - e^{-X}) - \tfrac{1}{4}[(b-1)^2 + a^2]\sin 2Z.$$

If the quartic also has a point of undulation, the corresponding surface is (∞^1 types):

$$Y = (b-1)\sin Z\,(e^X + e^{-X}) + 2\sqrt{b}\cos Z\,(e^X - e^{-X}) - \tfrac{1}{4}(b+1)^2\sin 2Z.$$

IV. *To a quartic with a triple point, two of whose tangents are coincident, there correspond ∞^1 types of surfaces*

$$Y = \tfrac{1}{3}X^3 - bX(e^Z + e^{-Z}).$$

V. *To a quartic which in addition to the triple point (two coincident tangents) also has a point of undulation there corresponds a single type of surfaces of the form*

$$Y = \tfrac{1}{3}X^3 - \tfrac{1}{4}X(e^Z + e^{-Z}).$$

The last two surfaces may also be put in the form

$$Y = -\tfrac{1}{3}X^3 - 2b\cos Z\,.\,X,$$
$$Y = -\tfrac{1}{3}X^3 - \tfrac{1}{2}X\cos Z.$$

VI. *If finally all the tangents at the triple point coincide, two types of algebraic surfaces are obtained which have been discussed in a former paper,* where the proof is given.*

We now give a résumé of the results obtained:

PLANE AT INFINITY.	SPACE (X, Y, Z).
I. Quartics with a triple point (real tangents).	a. $Y = b(e^{X-Z} - e^{Z-X}) - (a+b+1)(e^X - e^{-X}) + b(a+b+1)(e^Z - e^{-Z})$. b. $Y = b\sin(X-Z) - (a+b+1)\sin X + b(a+b+1)\sin Z$.
II. Quartics with a triple point and a point of undulation.	a. $Y = \frac{a^2}{4}(e^{X-Z} - e^{Z-X}) - \left(\frac{a}{2}+1\right)^2(e^X - e^{-X}) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2(e^Z - e^{-Z})$. b. $Y = \frac{a^2}{4}\sin(X-Z) + \left(\frac{a}{2}+1\right)^2\sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2\sin Z$.
III. Quartics having a triple point with one real and two imaginary tangents.	$Y = (b-1)\sin Z.(e^X + e^{-X}) + a\cos Z.(e^X - e^{-X}) - \frac{1}{4}[(b-1)^2 + a^2]\sin 2Z$.
IV. Quartics having a triple point with one real and two imaginary tangents and also a point of undulation.	$Y = (b-1)\sin Z(e^X + e^{-X}) + 2\sqrt{b}\cos Z(e^X - e^{-X}) - \frac{1}{4}(b+1)^2\sin 2Z$.
V. Quartics with a triple point, two of the tangents being coincident.	$Y = \frac{1}{3}X^3 - bX(e^Z + e^{-Z})$.
VI. Quartics with a triple point, two of the tangents being coincident, and having also a point of undulation.	$Y = \frac{1}{3}X^3 - \frac{1}{4}X(e^Z + e^{-Z})$.
VII. Quartics with a triple point, all three tangents being coincident.	Algebraic surface.
VIII. Quartic with a triple point, coincident tangents, and a point of undulation.	Algebraic surface.

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* *Am. Jour. of Math.*, vol. 29: On a Certain Class of Algebraic Translation-Surfaces, pp. 384–385.

Quartics with a Tac-Node and Double Point.

A quartic with a tac-node and a double point takes the following form after a proper projective transformation:

$$x^4 + cx^3y + dxy^2z + ay^2z^2 + bx^2yz = 0. \tag{1}$$

By means of an affinity transformation this curve may be thrown into the form $(z = 1)$:

$$x^4 + x^3y + x^2y + ay^2 + bxy^2 = 0,$$

which may be represented parametrically as follows:

$$x = -\frac{a + \rho + \rho^2}{\rho + b}, \quad y = \frac{(a + \rho + \rho^2)^2}{\rho(\rho + b)^2}.$$

We shall distinguish between 5 cases:

1. $a > \frac{1}{4}$, the tac-node is imaginary.
2. $a < \frac{1}{4}$, the tac-node is real.
3. $a = \frac{1}{4}$, ramphoid cusp and a node.
4. $b = 0$, tac-node and cusp (node either real or imaginary according as $a \lessgtr \frac{1}{4}$).
5. $a = \frac{1}{4}$, $b = 0$, ramphoid cusp and cusp.

The last case gives rise to an algebraic surface, as we have shown in a former paper, and will not be discussed here. We shall not go into the details of the calculations, only giving the chief results.

1. In the first case we obtain the surface

$$\begin{aligned} X &= \int \frac{d\rho_1}{a + \rho_1 + \rho_1^2} + \int \frac{d\rho_2}{a + \rho_2 + \rho_2^2}, \\ Y &= \log \frac{\rho_1 \rho_2}{(\rho_1 + b)(\rho_2 + b)}, \\ Z &= \int \frac{(\rho_1 + b)\, d\rho_1}{(a + \rho_1 + \rho_1^2)^2} + \int \frac{d\rho_2}{(a + \rho_2 + \rho_2^2)^2}, \end{aligned} \tag{2}$$

which, after integrating and transforming linearly, may be written $(a > \frac{1}{4})$:

$$\begin{aligned} X &= \tan^{-1} \frac{\rho_1 + \frac{1}{2}}{\sqrt{a - \frac{1}{4}}} + \tan^{-1} \frac{\rho_2 + \frac{1}{2}}{\sqrt{a - \frac{1}{4}}}, \\ Y &= \log \frac{\rho_1 \rho_2}{(\rho_1 + b)(\rho_2 + b)}, \\ Z &= \frac{\rho_1 + c}{a + \rho_1 + \rho_1^2} + \frac{\rho_2 + c}{a + \rho_2 + \rho_2^2}, \quad \left(c = \frac{b - 2a}{2b - 1}\right). \end{aligned} \tag{3}$$

Eliminating ρ_1 and ρ_2 from these equations we obtain a surface of the form

$$Z=\frac{A e^{2Y}\tan^2 X+B e^{2Y}+C\tan^2 X+De^{Y}\tan^2 X+Ee^{2Y}\tan X+Fe^{Y}\tan X+Ge^{Y}+H\tan X+I}{A' e^{2Y}\tan^2 X+B' e^{2Y}+C'\tan^2 X+D'e^{Y}\tan^2 X+E'e^{2Y}\tan X+F'e^{Y}\tan X+G'e^{Y}+H'\tan X+I'}. \quad (4)$$

We shall not endeavor to find the center of symmetry, as it involves very long calculations.

2. In the second case we get the same form as (4), only, instead of $\tan X$, we must substitute e^X.

3. Putting $a=\frac{1}{4}$ in (3) we obtain the surface

$$X=\frac{1}{\rho_1+\frac{1}{2}}+\frac{1}{\rho_2+\frac{1}{2}},$$

$$Y=\log\frac{\rho_1\rho_2}{(\rho_1+b)(\rho_2+b)},$$

$$Z=\frac{4}{(\rho_1+\frac{1}{2})^2}+\frac{2b-1}{(\rho_1+\frac{1}{2})^3}+\frac{4}{(\rho_2+\frac{1}{2})^2}+\frac{2b-1}{(\rho_2+\frac{1}{2})^3},$$

from which, by elimination of ρ_1 and ρ_2 and tranforming to the center of symmetry, we obtain a surface of the form

$$e^Y=\frac{A+BX+CX^2+DX^3+EZ}{A-BX+CX^2-DX^3-EZ}. \quad (5)$$

4. When $b=0$, we have the curve

$$x^4+x^3y+x^2y+ay^2=0,$$

and the corresponding surface may be written:

$$\begin{aligned} X&=\tan^{-1}\frac{\rho_1+{}^{1}}{\sqrt{a-\frac{1}{4}}}+\tan^{-1}\frac{\rho_2+\frac{1}{2}}{\sqrt{a-\frac{1}{4}}},\\ Y&=\frac{1}{\rho_1}+\frac{1}{\rho_2},\\ Z&=\frac{\rho_1+2a}{a+\rho_1+\rho_1^2}+\frac{\rho_2+2a}{a+\rho_2+\rho_2^2}; \end{aligned} \quad (6)$$

and in case $a<\frac{1}{4}$:

$$\begin{aligned} X&=\log\frac{(\rho_1-\sqrt{\frac{1}{4}-a})(\rho_2-\sqrt{\frac{1}{4}-a})}{(\rho_1+\sqrt{\frac{1}{4}-a})(\rho_2+\sqrt{\frac{1}{4}-a})},\\ Y&=\frac{1}{\rho_1}+\frac{1}{\rho_2},\\ Z&=\frac{\rho_1+2a}{a+\rho_1+\rho_1^2}+\frac{\rho_2+2a}{a+\rho_2+\rho_2^2}. \end{aligned} \quad (7)$$

From (6) we obtain a surface of the form

$$Z=\frac{AY^2\tan^2X+BY^2+C\tan^2X+DY\tan^2X+EY^2\tan X+FY\tan X+GY+H\tan X+I}{A'Y^2\tan^2X+B'Y^2+C'\tan^2X+D'Y\tan^2X+E'Y^2\tan X+F'Y\tan X+G'Y+H'\tan X+I'},$$

and from (7) *a surface of the same form,* e^X *being substituted for* $\tan X$.

Remark. If in the second case the node is imaginary, the curve may be put in a suitable form so that, in (5), $\tan Y$ will appear instead of e^Y.

Quartics with an Osc-Node.

This case has been discussed in my former paper on algebraic translation-surfaces* with a fourfold mode of representation. It was found that the surface may be represented parametrically as follows:

$$X=\int\frac{\left(\rho_1+\frac{c}{2}\right)d\rho_1}{\rho^4-\left(\frac{c^2}{4}-1\right)^2}+\int\frac{\left(\rho_2+\frac{c}{2}\right)d\rho_2}{\rho_2^4-\left(\frac{c^2}{4}-1\right)^2},$$

$$Y=\int\frac{d\rho_1}{\rho_1^2-\left(\frac{c^2}{4}-1\right)}+\int\frac{d\rho_2}{\rho_2^2-\left(\frac{c^2}{4}-1\right)},$$

$$Z=\int\frac{\left(c\rho_1+\frac{c^2}{2}-1\right)d\rho_1}{\left(\rho_1^2+\frac{c^2}{4}-1\right)^2\left[\rho_1^2-\left(\frac{c^2}{4}-1\right)\right]}+\int\frac{\left(c\rho_2+\frac{c^2}{2}-1\right)d\rho_2}{\left(\rho_2^2+\frac{c^2}{4}-1\right)\left[\rho_2^2-\left(\frac{c^2}{4}-1\right)\right]},$$

which we shall not discuss in detail. When $c=\pm 2$, we obtain an algebraic surface which has been treated in the previous paper. The classification of translation-surfaces connected with a unicursal quartic is thus completed. The cubic surface obtained on p. 178 deserves a closer study inasmuch as, from the standpoint of the theory of translation-surfaces, it holds a unique place in geometry. The existence and properties of such a surface fully realize the expectation of Georg Scheffers when he says (*Acta Math.*, Vol. XXVIII, 1904, p. 90): "Die grosse Zahl verschiedenartiger Typen von Translationsflächen, die sich aus dem Lie'schen Theorem ergeben, ist bisher, so viel ich weiss, noch nicht genauer untersucht worden, obgleich ihre Betrachtung wegen des innigen Zusammenhangs mit dem Abel'schen Theorem sowohl in geometrischer als auch in analytischer Hinsicht gewiss sehr lohnend sein würde."

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* On a Certain Class, etc.: *Am. Jour. of Math.* vol. 29, p. 382.

The interpretation of this and the previous paper from the standpoint of the theory of functions is so evident that we have not thought it worth while to dwell on it. The functions obtained are all *degenerate Abelian Integrals of the second and third kind*[*] (polar and logarithmic singularities).

UNIVERSITY OF WEST VIRGINIA, *July* 20, 1907.

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The Determination of the Conjugate Points for Discontinuous Solutions in the Calculus of Variations.

BY OSKAR BOLZA.

In §§8 and 9 of his Inaugural-Dissertation, "*Ueber die discontinuierlichen Lösungen in der Variationsrechnung*" (Göttingen, 1904), Caratheodory develops the general theory of the conjugate points for discontinuous solutions. The object of the present note is to derive Caratheodory's results concerning conjugate points by a more direct method, to supplement them in certain points, and to give in particular, in explicit form, the equation which connects the parameters of a pair of conjugate points.

§ 1. *Sets of "Broken Extremals".*

In order that a curve $P_1 P_0 P_2$ with a "corner" at P_0, but otherwise of class* C', may minimize† the integral

$$J = \int_{t_1}^{t_2} F(x, y, x', y')\, dt,$$

it is in the first place necessary that the two "continuous" branches $P_1 P_0$ and $P_0 P_2$ should separately satisfy the four necessary conditions for a minimum with fixed endpoints. In particular, each one of the two arcs $P_1 P_0$ and $P_0 P_2$ must be an *extremal.*

Further, at the point $P_0 (x_0, y_0)$ Weierstrass-Erdmann's *corner-condition*‡ must be satisfied:

$$\begin{aligned} F_{x'}(x_0, y_0, \cos\vartheta_0, \sin\vartheta_0) &= F_{x'}(x_0, y_0, \cos\bar{\vartheta}_0, \sin\vartheta_0), \\ F_{y'}(x_0, y_0, \cos\vartheta_0, \sin\vartheta_0) &= F_{y'}(x_0, y_0, \cos\bar{\vartheta}_0, \sin\bar{\vartheta}_0), \end{aligned} \qquad (1)$$

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* Compare for the terminology my *Lectures on the Calculus of Variations*, § 2, c) and § 24, a).

† In the sense defined in § 24, c) of my *Lectures* and under the assumptions concerning the function $F(x, y, x', y')$ stated in § 24, b).

‡ Compare *Lectures*, § 25, c).

where ϑ_0 denotes the amplitude of the positive tangent to the arc $P_1 P_0$ at P_0, $\bar{\vartheta}_0$ the amplitude of the positive tangent to the arc $P_0 P_2$ at P_0.

We shall call a curve $P_1 P_0 P_2$ consisting of two arcs of extremals $P_1 P_0$ and $P_0 P_2$ a *"broken extremal"*, if at P_0 this corner-condition (1) is satisfied.

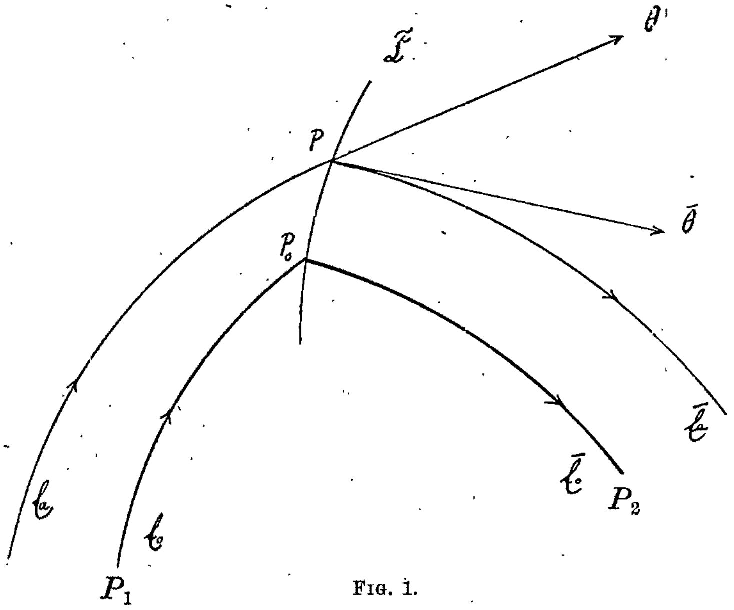

FIG. 1.

We assume for the following discussion that the curve $P_1 P_0 P_2$ lies in the *interior* of the domain of continuity R of the function F (compare *Lectures*, § 24, b), and that Legendre's condition is satisfied in the stronger form*

$$F_1 > 0 \tag{2}$$

along each of the two branches $P_1 P_0$ and $P_0 P_2$.

Let now

$$x = \phi(t, a), \quad y = \psi(t, a) \tag{3}$$

be any one-parameter set of extremals which contains the arc $P_1 P_0$ for $a = a_0$, so that the arc $P_1 P_0$ is representable by the equations

$$x = \phi(t, a_0), \quad y = \psi(t, a_0), \quad t_1 \leqq t \leqq t_0. \tag{4}$$

The functions

$$\phi, \phi_t, \phi_{tt}; \quad \psi, \psi_t, \psi_{tt}$$

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* Compare *Lectures*, § 27, b).

are supposed* to be of class C' as functions of t and a in the domain

$$t_1 - h_1 \overset{=}{<} t \overset{=}{<} t_0 + h_0, \quad | a - a_0 | \overset{=}{<} d,$$

h_0, h_1, d being sufficiently small positive quantities.

The extremal of the set (3) corresponding to a particular value a will be denoted by $\mathfrak{E}_a$; further we write $\mathfrak{E}_0$ for $\mathfrak{E}_{a_0}$.

We propose to determine a point $P(t)$ on a given extremal $\mathfrak{E}_a$ of the set (3), and at the same time a direction $\bar{\vartheta}$ passing through P, such that the direction $\bar{\vartheta}$ together with the direction ϑ of the positive tangent to the extremal $\mathfrak{E}_a$ at P shall satisfy Weierstrass-Erdmann's corner-condition for the point P.

We have, then, for the determination of the two unknown quantities t and $\bar{\vartheta}$, the two equations†:

$$\begin{aligned} &F_{x'}[\phi(t,a), \psi(t,a), \phi_t(t,a), \psi_t(t,a)] - F_{x'}[\phi(t,a), \psi(t,a), \cos\bar{\vartheta}, \sin\bar{\vartheta}] = 0, \\ &F_{y'}[\phi(t,a), \psi(t,a), \phi_t(t,a), \psi_t(t,a)] - F_{y'}[\phi(t,a), \psi(t,a), \cos\bar{\vartheta}, \sin\bar{\vartheta}] = 0. \end{aligned} \tag{5}$$

These equations are satisfied for $t = t_0$, $a = a_0$, $\bar{\vartheta} = \bar{\vartheta}_0$, since according to our assumptions the broken extremal $P_1 P_0 P_2$ satisfies the corner-condition (1). Further, their left-hand members, which we denote by $\Phi(t, a, \bar{\vartheta})$ and $\Psi(t, a, \bar{\vartheta})$ respectively, are of class C' in the vicinity of the point t_0, a_0, $\bar{\vartheta}_0$. Hence we can apply the theorem on implicit functions,‡ provided that the Jacobian

$$J_{t\bar{\vartheta}} = \frac{\partial(\Phi, \Psi)}{\partial(t, \bar{\vartheta})}$$

is different from zero at the point t_0, a_0, $\bar{\vartheta}_0$. If we write for brevity

$$\cos\vartheta = p, \ \sin\vartheta = q; \ \cos\vartheta = \bar{p}, \ \sin\vartheta = \bar{q},$$

and remember that along the extremal $P_1 P_0$

$$\frac{\partial}{\partial t} F_{x'} = F_x, \quad \frac{\partial F_{y'}}{\partial t} = F_y,$$

we obtain:

$$\begin{aligned} \Phi_t &= F_x - \bar{F}_{x'x}\, x' - \bar{F}_{x'y}\, y', & \Psi_t &= F_y - \bar{F}_{y'x}\, x' - \bar{F}_{y'y}\, y', \\ \Phi_{\bar{\vartheta}} &= \bar{F}_1 \bar{q}, & \Psi_{\bar{\vartheta}} &= -\bar{F}_1 \bar{p}, \end{aligned} \tag{6}$$

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* The existence of an infinitude of sets of extremals satisfying these conditions is a consequence of our assumptions according to certain existence theorems on differential equations; compare Kneser, *Lehrbuch der Variationsrechnung*, § 27, and Bolza, *Trans. Amer. Math. Soc.*, Vol. VII (1906), p. 464.

† Since $F_{x'}$, $F_{y'}$ are positively homogeneous of dimension zero in x', y', we may replace in these functions $\cos\vartheta$, $\sin\vartheta$ by $\phi_t(t, a)$, $\psi_t(t, a)$.

‡ Compare, for instance, Osgood, *Lehrbuch der Functionentheorie*, Vol. I, p. 52.

where the arguments of F_x, F_y are: $\phi(t,a)$, $\psi(t,a)$, $x' = \phi_t(t,a)$, $y' = \psi_t(t,a)$; those of $\bar F_1$, $\bar F_{x'x}$, etc.: $\phi(t,a)$, $\psi(t,a)$, $\bar p$, $\bar q$.

Making use of the homogeneity properties* of the function F and its partial derivatives, we obtain for the above Jacobian:

$$J_{t\bar\vartheta} = \sqrt{x'^2+y'^2}\,\bar F_1 \{ p\bar F_x + q\bar F_y - (\bar p F_x + \bar q F_y) \}, \tag{7}$$

where now the two last arguments in F_x, F_y are p, q.

The first two factors of $J_{t\bar\vartheta}$ are different from zero for $t = t_0$, $a = a_0$, $\bar\vartheta = \bar\vartheta_0$. Hence if we put, with Caratheodory,

$$\begin{aligned} \Omega(x_0, y_0) = p_0 F_x(x_0, y_0, \bar p_0, \bar q_0) + q_0 F_y(x_0, y_0, \bar p_0, \bar q\,) \\ - \bar p_0 F_x(x_0, y_0, p_0, q_0) - \bar q_0 F_y(x_0, y_0, p_0, q_0), \end{aligned} \tag{8}$$

where

$$p_0 = \cos\vartheta_0,\quad q_0 = \sin\vartheta_0,\quad \bar p_0 = \cos\bar\vartheta_0,\quad \bar q_0 = \sin\bar\vartheta_0,$$

we have the result:

If the condition

$$\Omega(x_0, y_0) \neq 0 \tag{9}$$

is satisfied, there exists one and but one system of functions

$$t = t(a), \quad \bar\vartheta = \bar\vartheta(a), \tag{10}$$

of class C' in the vicinity of $a = a_0$, which satisfies the two equations (5) *and the initial conditions*

$$t(a_0) = t_0, \quad \bar\vartheta(a_0) = \bar\vartheta_0. \tag{11}$$

The functions (10) represent, at least for the vicinity of the point P_0, the solution of the problem proposed above.

From our assumption (2), applied to the point P_0 and the branch $P_0 P_2$, it follows that

$$F_1(\phi[t(a), a],\ \psi[t(a), a],\ \cos\bar\vartheta(a),\ \sin\bar\vartheta(a)) \neq 0$$

for all sufficiently small values of $|a - a_0|$. Hence† it is possible to construct one and but one extremal

$$\bar{\mathfrak{E}}_a:\ x = \bar\phi(t,a),\quad y = \bar\psi(t,a) \tag{12}$$

through the point P in the direction $\bar\vartheta(a)$. The parameter t can be so selected that also on $\bar{\mathfrak{E}}_a$ the value $t = t(a)$ furnishes the point P, so that

$$\bar\phi[t(a), a] = \phi[t(a), a],\quad \bar\psi[t(a), a] = \psi[t(a), a]. \tag{13}$$

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* Compare *Lectures*, § 24, b) equations (8) and (10).

† According to Cauchy's existence theorem on differential equations; compare *Lectures*, § 25, b).

We thus obtain a broken extremal $\mathfrak{E}_a + \bar{\mathfrak{E}}_a$ with a corner at P, on which the parameter t varies continuously. If we let a vary, we obtain a *set of broken extremals*. We shall call the set (12) the *set of extremals complementary* to the set (3). On account of (11) it contains, for $a = a_0$, the extremal $\bar{\mathfrak{E}}_0$ of which the arc $P_0 P_2$ forms a part.

From the properties of the integrals of a system of differential equations as functions of their initial values,* it follows that the functions $\bar{\phi}(t, a)$, $\bar{\psi}(t, a)$ have the same continuity properties as the functions $\phi(t, a)$, $\psi(t, a)$, in a domain

$$t_0 - \bar{h}_0 \leqq t \leqq t_2 + h_2, \quad | a - a_0 | \leqq \bar{d}.$$

§ 2. *The Corner-Curve.*†

If we let a vary, the corner P describes a curve $\tilde{\mathfrak{C}}$, which we call the *"corner-curve"*. If we define the functions $\tilde{x}(a)$, $\tilde{y}(a)$ by the equations

$$\tilde{x}(a) = \phi\,[t(a), a], \quad \tilde{y}(a) = \psi\,[t(a), a], \tag{14}$$

or, what amounts to the same thing according to (13),

$$\tilde{x}(a) = \bar{\phi}\,[t(a), a], \quad \tilde{y}(a) = \bar{\psi}\,[t(a), a], \tag{14a}$$

the corner-curve is given in parameter-representation by the equations

$$\tilde{\mathfrak{C}}: \quad x = \tilde{x}(a), \quad y = \tilde{y}(a),$$

and any particular value of a furnishes that point of $\tilde{\mathfrak{C}}$ which is the corner for the corresponding broken extremal $\mathfrak{E}_a + \bar{\mathfrak{E}}_a$.

We propose first to compute the slope $\tan \tilde{\vartheta}$ of the tangent to the corner-curve $\tilde{\mathfrak{C}}$ at the point P.

From the definition of the functions $\tilde{x}$, $\tilde{y}$, we obtain for their derivatives with respect to a:

$$\tilde{x}' = \phi_t\, t'(a) + \phi_a, \quad \tilde{y}' = \psi_t\, t'(a) + \psi_a;$$

and from (5) we obtain, according to the rules for the differentiation of implicit functions,

$$t'(a) = -\frac{J_{a\bar{\vartheta}}}{J_{t\bar{\vartheta}}},$$

where

$$J_{a\bar{\vartheta}} = \frac{\partial(\Phi, \Psi)}{\partial(a, \bar{\vartheta})}.$$

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* Compare Kneser, *Lehrbuch der Variationsrechnung*, § 27, and Bliss, The Solution of Differential Equations of the First Order as Functions of their Initial Values, *Annals of Mathematics*, Ser. 2, Vol. VI, p. 49.

† Caratheodory's "Knickpunkt-Curve".

But

$$\Phi_a = F_{x'x}\phi_a + F_{x'y}\psi_a + F_{x'x'}\phi_{ta} + F_{x'y'}\psi_{ta} - \bar{F}_{x'x}\phi_a - \bar{F}_{x'y}\psi_a,$$
$$\Psi_a = F_{y'x}\phi_a + F_{y'y}\psi_a + F_{y'x'}\phi_{ta} + F_{y'y'}\psi_{ta} - \bar{F}_{y'x}\phi_a - \bar{F}_{y'y}\psi_a;$$

the functions $\bar{F}_{x'x}$, $\bar{F}_{x'y}$, $\bar{F}_{y'x}$, $\bar{F}_{y'y}$ are positively homogeneous of dimension zero with respect to their last two arguments $\bar{p}, \bar{q}$; hence we may replace $\bar{p}$ and $\bar{q}$ by $\phi_t(t, a)$ and $\bar{\psi}_t(t, a)$ respectively. This being done, we express all the partial derivatives of F in terms of Weierstrass' functions*: F_1, L, M, N. The result is

$$\begin{aligned}\Phi_a &= -A\phi_a - B\psi_a - y'\Delta_t F_1 - \bar{y}'\bar{F}_1(\phi_a\bar{y}'' - \psi_a\bar{x}''),\\ \Psi_a &= -B\phi_a - C\psi_a + x'\Delta_t F_1 + \bar{x}'\bar{F}_1(\phi_a\bar{y}'' - \psi_a\bar{x}''),\end{aligned} \tag{15}$$

where

$$x' = \phi_t(t, a),\ y' = \psi_t(t, a);\ \bar{x}' = \bar{\phi}_t(t, a),\ \bar{y}' = \bar{\psi}_t(t, a);$$
$$\bar{x}'' = \bar{\phi}_{tt}(t, a),\ \bar{y}'' = \bar{\psi}_{tt}(t, a);\ \Delta(t, a) = \phi_t\psi_a - \psi_t\phi_a,$$
$$A = \bar{L} - L,\ B = \bar{M} - M,\ C = \bar{N} - N;$$

the quantities L, M, N refer to the point P and the extremal $\mathfrak{E}_a$, the quantities $\bar{L}, \bar{M}, \bar{N}$ to the point P and the extremal $\bar{\mathfrak{E}}_a$. Finally, the last two arguments of F_1 and $\bar{F}_1$ are x', y' and $\bar{x}'$, $\bar{y}'$ respectively.

From (15) and (6) we obtain

$$J_{a\bar{\vartheta}} = (\bar{x}'^2 + \bar{y}'^2)\bar{F}_1\{\phi_a(A\bar{x}' + B\bar{y}') + \psi_a(B\bar{x}' + C\bar{y}') - \Delta_t F_1(x'\bar{y}' - y'\bar{x}')\}. \tag{16}$$

At the same time the expression (7) for $J_{t\bar{\vartheta}}$ may be thrown into another form, if we remember the homogeneity properties of F_1, F_x, F_y and make use of the relations†

$$Lx' + My' = F_x, \quad Mx' + Ny' = F_y;$$

we thus obtain

$$J_{t\bar{\vartheta}} = (\bar{x}'^2 + \bar{y}'^2)\bar{F}_1[Ax'\bar{x}' + B(x'\bar{y}' + y'\bar{x}') + Cy'\bar{y}']. \tag{17}$$

The comparison between the two expressions for $J_{t\bar{\vartheta}}$ leads to a second form for the quantity $\Omega(x, y)$; viz.,

$$\Omega(x, y) = Ap\bar{p} + B(p\bar{q} + q\bar{p}) + Cq\bar{q}. \tag{18}$$

We thus finally obtain

$$\begin{aligned}\tilde{x}' &= \frac{-\Delta(B\bar{x}' + C\bar{y}') + x'\Delta_t F_1(x'\bar{y}' - y'\bar{x}')}{Ax'\bar{x}' + B(x'\bar{y}' + y'\bar{x}') + Cy'\bar{y}'},\\ \tilde{y}' &= \frac{\Delta(A\bar{x}' + B\bar{y}') + y'\Delta_t F_1(x'\bar{y}' - y'\bar{x}')}{Ax'\bar{x}' + B(x'\bar{y}' + y'\bar{x}') + Cy'\bar{y}'}.\end{aligned} \tag{19}$$

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* Compare *Lectures*, Chap. IV, equations (11 a) and (35).

† Compare *Lectures*, p. 132.

Hence follows, *for the slope* $\tan\tilde{\vartheta}$ *of the tangent to the corner-curve* $\tilde{\mathfrak{C}}$ *at the point* P, the expression

$$\tan\tilde{\vartheta}=\frac{\Delta\,(A\,\bar{x}'+B\,\bar{y}')+y'\,\Delta_t\,F_1\,(x'\,\bar{y}'-y'\,\bar{x}')}{-\,\Delta\,(B\,\bar{x}'+C\,\bar{y}')+x'\,\Delta_t\,F_1\,(x'\,\bar{y}'-y'\,\bar{x}')}\,. \tag{20}$$

§ 3. *Interrelation Between the Slope of the Corner-Curve at* P_0 *and the Focal-Points of the Set of Broken Extremals.*

We now consider in particular the question how the slope $\tan\tilde{\vartheta}_0$ of the tangent to the corner-curve at P_0 depends upon the choice of the set of extremals (3). For this purpose we have to put $a=a_0$ in (20), and consequently, according to (11), the argument $t=t(a)$, in x', y'; $\bar{x}'$, $\bar{y}'$, $\Delta\,(t, a)$, etc., equal to t_0. In the resulting expression for $\tan\tilde{\vartheta}_0$, the Jacobian $\Delta\,(t_0, a_0)$ and its derivative $\Delta_t\,(t_0, a_0)$ are the only quantities which depend upon the choice of the set of extremals (3).

The function $\Delta\,(t, a_0)$, in its turn, is determined to a constant factor by the condition that it satisfies Jacobi's differential equation* for the extremal $\mathfrak{E}_0$, viz.,

$$F_2\,u-\frac{d}{dt}\left(F_1\,\frac{du}{dt}\right)=0, \tag{21}$$

and by one of its zeros. Let $t=\tau$ be the zero of $\Delta\,(t, a_0)$ next smaller than t_0, so that the corresponding point of $\mathfrak{E}_0$, which we denote by Q, is the focal point† of the set (3) on $\mathfrak{E}_0$. Then

$$\Delta\,(t, a_0)=\text{Const.}\,\Theta\,(t, \tau),$$

where $\Theta\,(t, \tau)$ is the function which determines in Weierstrass'‡ theory the conjugate point to Q. We may therefore write

$$\tan\tilde{\vartheta}_0=\frac{\alpha\,\Theta\,(t_0, \tau)+\beta\,\Theta_t\,(t_0, \tau)}{\gamma\,\Theta\,(t_0, \tau)+\delta\,\Theta_t\,(t_0, \tau)}, \tag{22}$$

where

$$\begin{aligned}\alpha&=A_0\,\bar{p}_0+B_0\,\bar{q}_0, & \beta&=q_0\,F_1\,(t_0)\sin\,(\vartheta_0-\vartheta_0)\,(x_0'^2+y_0'^2),\\ \gamma&=-\,(B_0\,\bar{p}_0+C_0\,\bar{q}_0), & \delta&=p_0\,F_1\,(t_0)\sin\,(\bar{\vartheta}_0-\vartheta_0)\,(x_0'^2+y_0'^2),\end{aligned} \tag{23}$$

the subscript 0 indicating that the quantities to which it is affixed are to be computed for the point P_0.

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* Compare *Lectures*, pp. 40 and 200.

† Compare Kneser, *Lehrbuch der Variationsrechnung*, § 24, and my *Lectures*, § 38.

‡ Compare *Lectures*, p. 135.

The coefficients α, β, γ, δ are therefore independent of τ. Hence *the slope of the corner-curve* $\tilde{\mathfrak{C}}$ *at* P_0 *is the same for all sets of extremals* (3) *which have the same focal point* Q, the set of extremals through the point Q being included among the latter.

We examine next how the slope $\tan \tilde{\vartheta}_0$ varies when the focal point Q describes the extremal $\mathfrak{E}_0$. For this purpose, we compute the derivative of $\tan \tilde{\vartheta}_0$ with respect to τ:

$$\frac{d \tan \tilde{\vartheta}_0}{d\tau} = - \frac{(\alpha \delta - \beta \gamma) \{ \Theta(t_0, \tau) \Theta_{t\tau}(t_0, \tau) - \Theta_t(t_0, \tau) \Theta_\tau(t_0, \tau) \}}{\{ \gamma \Theta(t_0, \tau) + \delta \Theta_t(t_0, \tau) \}^2} .$$

But from the definition of $\Theta(t, \tau)$ it follows that

$$\begin{aligned} &\Theta(t_0, \tau) \Theta_{t\tau}(t_0, \tau) - \Theta_t(t_0, \tau) \Theta_\tau(t_0, \tau) \\ &\qquad = [\vartheta_1(t_0) \vartheta_2'(t_0) - \vartheta_2(t_0) \vartheta_1'(t_0)] [\vartheta_1(\tau) \vartheta_2'(\tau) - \vartheta_2(\tau) \vartheta_1'(\tau)], \end{aligned}$$

where $\vartheta_1(t)$, $\vartheta_2(t)$ are two linearly independent solutions of Jacobi's differential equation (21). Hence from the theory of linear differential equations it follows* that

$$\vartheta_1(t) \vartheta_2'(t) - \vartheta_2(t) \vartheta_1'(t) = \frac{k}{F_1(t)},$$

where k is a constant different from zero.

On the other hand we get, on substituting the values of α, β, γ, δ,

$$a \delta - \beta \gamma = F_1(t_0) \sin(\vartheta_0 - \vartheta_0) \Omega(x_0, y_0) (x_0'^2 + y_0'^2).$$

Hence it follows that

$$\frac{d}{d\tau} \tan \tilde{\vartheta}_0 = \frac{-k^2 (x_0'^2 + y_0'^2) \sin(\bar{\vartheta}_0 - \vartheta_0) \Omega(x_0, y_0)}{F_1(\tau) \{ \gamma \Theta(t_0, \tau) + \delta \Theta_t(t_0, \tau) \}^2} . \tag{24}$$

We suppose for the further discussion that

$$\vartheta_0 - \vartheta_0 \not\equiv 0 \pmod{\pi}, \tag{25}$$

and that the inequality (2) holds not only for the arc $P_1 P_0$ of the extremal $\mathfrak{E}_0$ but also for its continuation beyond P_1, at least as far as the point P_0' $(t = t_0')$ whose conjugate the point P_0 is.

And now we let τ increase from t_0' to t_0; *i. e.*, we let the point Q describe the extremal $\mathfrak{E}_0$ from P_0' to P_0. The derivative of $\tan \tilde{\vartheta}_0$ will then always have a constant sign, since $\Omega(x_0, y_0)$, which is independent of τ, is supposed to be different from zero. For $\tau = t_0'$ and $\tau = t_0$, but for no other value between them, the function $\Theta(t_0, \tau)$ vanishes and $\tan\tilde{\vartheta}_0$ takes the value

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* Compare, for instance, *Lectures*, p. 58, footnote ².

$$\tan\tilde{\vartheta}_0 = \frac{\beta}{\delta} = \frac{q_0}{p_0} = \tan\vartheta_0.$$

Hence we have the result:

While the point Q describes the extremal $\mathfrak{E}_0$ from P_0' to P_0, the line $\tilde{\vartheta}_0$ revolves about the point P_0 from the initial position ϑ_0 constantly in the same sense through an angle of* 180°. The rotation takes place:

In positive sense, when $\Omega(x_0, y_0)\sin(\bar{\vartheta}_0 - \vartheta_0) < 0$;
In negative sense, when $\Omega(x_0, y_0)\sin(\bar{\vartheta}_0 - \vartheta_0) > 0$.

It passes therefore once and but once through the position $\bar{\vartheta}_0$. We denote the value of τ for which this takes place by e_0 and the corresponding point† of $\mathfrak{E}_0$ by E_0. For the discussion of sufficient conditions, it is important to distinguish whether the line $\tilde{\vartheta}_0$ lies in the angle‡ between the two branches P_1P_0 and P_0P_2 or outside of it. Four cases must be distinguished according to the signs of $\Omega(x_0, y_0)$ and $\sin(\bar{\vartheta}_0 - \vartheta_0)$. The result is:

While the point Q moves from P_0' to E_0, the line $\tilde{\vartheta}_0$ revolves from the position ϑ_0 into the position $\bar{\vartheta}_0$, inside of the angle between P_1P_0 and P_0P_2 when $\Omega(x_0, y_0) > 0$, outside of it when $\Omega(x_0, y_0) < 0$. As the point Q moves on from E_0 to P_0, the line $\tilde{\vartheta}_0$ continues its rotation from the position $\bar{\vartheta}_0$ into the position ϑ_0, outside of the angle in question when $\Omega(x_0, y_0) > 0$, inside of it when $\Omega(x_0, y_0) < 0$.

Conversely: To every line $\tilde{\vartheta}_0$ through the point P_0 which is tangent to neither of the two arcs P_1P_0, P_0P_2 at P_0, there belongs one and but one point Q, between P_0' and P_0, such that the corner-curve for every set of extremals (3) for which Q is the focal point, touches the line $\tilde{\vartheta}_0$ at P_0.

The value of τ belonging to a given line $\tilde{\vartheta}_0$ is obtained by solving equation (22) with respect to τ. The equation may be thrown into the form

$$\begin{aligned}&[A_0\bar{p}_0\tilde{p}_0 + B_0(\bar{p}_0\tilde{q}_0 + \bar{q}_0\tilde{p}_0) + C_0\bar{q}_0\tilde{q}_0]\,\Theta(t_0, \tau)\\ &\qquad - (x_0'^2 + y_0'^2)F_1(t_0)\sin(\bar{\vartheta}_0 - \vartheta_0)\sin(\tilde{\vartheta}_0 - \vartheta_0)\,\Theta_t(t_0', \tau) = 0,\end{aligned}\tag{26}$$

where

$$\tilde{p}_0 = \cos\tilde{\vartheta}_0, \quad \tilde{q}_0 = \sin\tilde{\vartheta}_0.$$

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* *I. e.*, the line through P_0 of slope $\tan\tilde{\vartheta}_0$.

† Caratheodory denotes this point by E_1; see *Dissertation*, p. 31.

‡ *I. e.*, that one of the two angles formed by the half-rays $\bar{\vartheta}_0$ and $\vartheta_0 + \pi$ which is less than π.

Case I: $\sin(\bar{\vartheta}_0 - \vartheta_0) > 0$, $\Omega(x_0, y_0) > 0$.

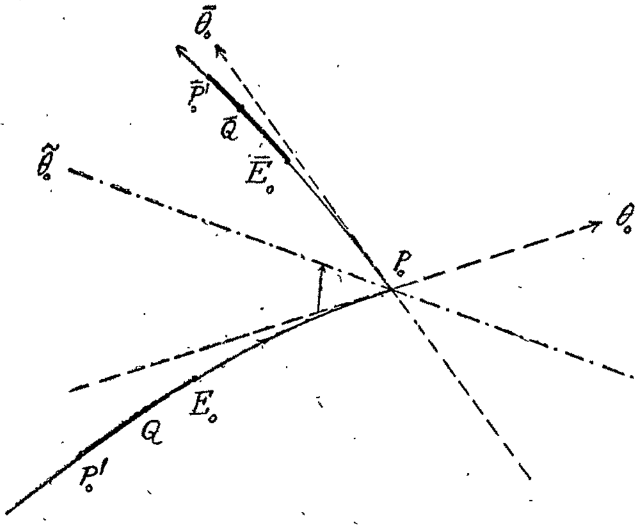

FIG. 2.

Case II: $\sin(\bar{\vartheta}_0 - \vartheta_0) > 0$, $\Omega(x_0, y_0) < 0$.

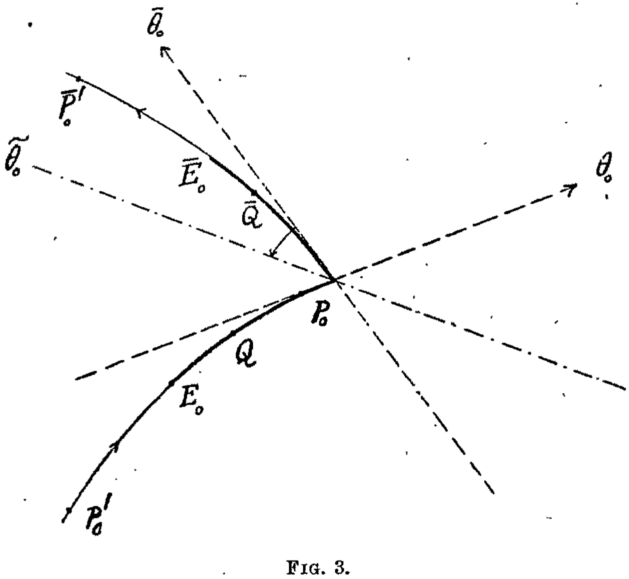

FIG. 3.

Case III: $\sin(\bar{\vartheta}_0 - \vartheta_0) < 0$, $\Omega(x_0, y_0) > 0$.

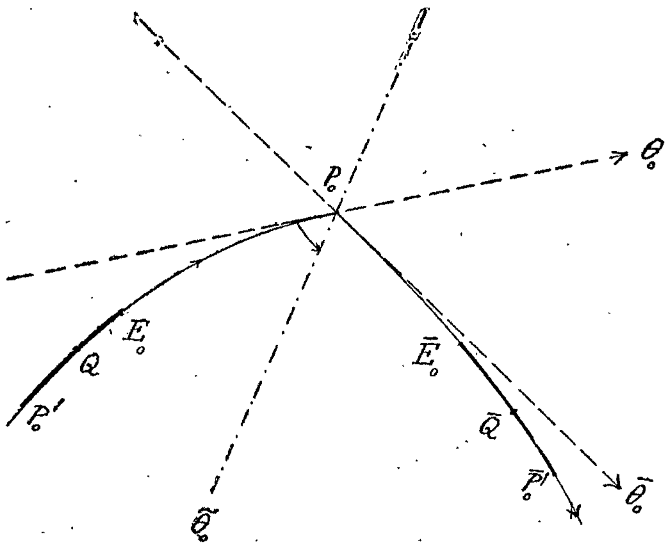

FIG. 4.

Case IV: $\sin(\bar{\vartheta}_0 - \vartheta_0) < 0$, $\Omega(x_0, y_0) < 0$.

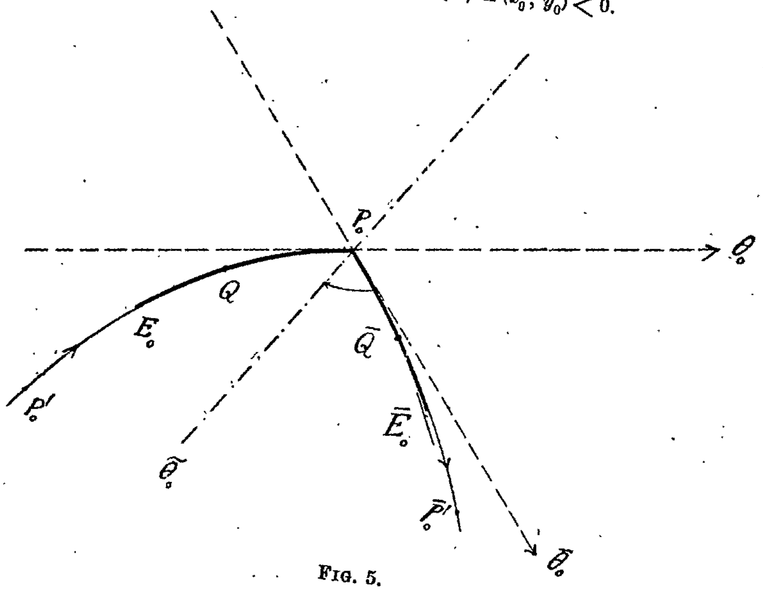

FIG. 5.

In particular, the equation for the determination of the parameter e_0 of the point E_0 is obtained by putting, in (26), $\tilde{\vartheta}_0 = \vartheta_0$.

§ 4. *The Conjugate Points of Discontinuous Solutions.*

Let, for a moment, the equations (12) represent any set of extremals containing, for $a = a_0$, the extremal $\bar{\mathfrak{E}}_0$. We may then propose for the set (12) the same problem which we have solved in § 1 for the set (3). The only difference will be that in the equations (5) the symbols ϕ, ψ, $\tilde{\vartheta}$ must be interchanged with $\bar{\phi}$, $\bar{\psi}$, ϑ, and the same interchange must be applied in the results; in this process the quantities A, B, C are changed into $-A$, $-B$, $-C$. Accordingly, if $\bar{Q}\,(t = \bar{\tau})$ be the focal point of the set (12) on $\bar{\mathfrak{E}}_0$, the slope of the corner-curve belonging to the set (12) at P_0 is

$$\tan \bar{\bar{\vartheta}}_0 = \frac{\bar{\alpha}\,\bar{\Theta}\,(t_0, \bar{\tau}) + \bar{\beta}\,\bar{\Theta}_t\,(t_0, \bar{\tau})}{\bar{\gamma}\,\bar{\Theta}\,(t_0, \bar{\tau}) + \bar{\delta}\,\bar{\Theta}_t\,(t_0, \bar{\tau})}, \tag{27}$$

where the quantities $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are derived from $\alpha, \beta, \gamma, \delta$ by the above interchange, and $\bar{\Theta}$ has the same meaning for $\bar{\mathfrak{E}}_0$ as Θ for $\mathfrak{E}_0$.

Conversely, we obtain the value of $\bar{\tau}$ corresponding to a given line $\bar{\bar{\vartheta}}$ by solving equation (27). We denote the value of $\bar{\tau}$ corresponding to the particular line ϑ_0 by $\bar{e}_0$ and the corresponding point of $\bar{\mathfrak{E}}_0$ by $\bar{E}_0$; this point lies between the point P_0 and its conjugate $\bar{P}_0'\,(t = \bar{t}_0')$ on $\bar{\mathfrak{E}}_0$.

Let now the equation (12) denote again, as in § 1, the particular set of extremals complementary to the set (3). The two sets (3) and (12) will then have the corner-curve in common; hence we have, in this case,

$$\bar{\bar{\vartheta}}_0 = \tilde{\vartheta}_0 .$$

We obtain, therefore, the focal point of the set (12) complementary to the set (3) by equating the right-hand members of the two equations (22) and (27) and solving the equation thus obtained with respect to τ. After some reductions the following result is obtained:

If $t = \tau$ be the parameter of the focal point Q of the set of extremals (3) *on $\mathfrak{E}_0$, and $t = \bar{\tau}$ the parameter of the focal point $\bar{Q}$ of the set* (12), *complementary to* (3), *on $\bar{\mathfrak{E}}_0$, then the following relation holds between τ and $\bar{\tau}$:*

$$\left.\begin{aligned}&(A_0 C_0 - B_0^2)\,\Theta(t_0,\tau)\,\overline{\Theta}(t_0,\bar{\tau})\\ &\qquad - (x_0'^2 + y_0'^2)\,F_1(t_0)\,(A_0 p_0^2 + 2\,B_0 p_0 q_0 + C_0 q_0^2)\,\frac{\partial\,\Theta(t_0,\tau)}{\partial\,t_0}\,\overline{\Theta}(t_0,\tau)\\ &\qquad + (\bar{x}_0'^2 + \bar{y}_0'^2)\,\overline{F}_1(t_0)\,(A_0\,\bar{p}_0^2 + 2\,B_0\,\bar{p}_0\,\bar{q}_0 + C_0\,\bar{q}_0^2)\,\Theta(t_0,\tau)\,\frac{\partial\,\overline{\Theta}}{\partial\,t_0}(t_0,\tau)\\ &- (x_0'^2 + y_0'^2)(\bar{x}_0'^2 + \bar{y}_0'^2)\,F_1(t_0)\,\overline{F}_1(t_0)\,\sin^2(\bar{\vartheta}_0 - \vartheta_0)\,\frac{\partial\,\Theta(t_0,\tau)}{\partial\,t_0}\,\frac{\partial\,\overline{\Theta}(t_0,\bar{\tau})}{\partial\,t_0} = 0.\end{aligned}\right\}\quad(28)$$

The two points Q and $\overline{Q}$ are called, according to Caratheodory,* *a pair of conjugate points of the broken extremal* $\mathfrak{E}_0 + \overline{\mathfrak{E}}_0$. According to a previous remark, the point $\overline{Q}$ conjugate to Q on $\mathfrak{E}_0 + \overline{\mathfrak{E}}_0$ may also be defined as *the focal point on* $\overline{\mathfrak{E}}_0$ *of the set of extremals which is complementary to the set of extremals through the point* Q.

In Figs. 2 to 5 the interrelation between the points Q and $\overline{Q}$ and the line $\tilde{\vartheta}_0$ is indicated. For instance, in Case I the point $\overline{Q}$ moves on $\overline{\mathfrak{E}}_0$ from $\overline{E}_0$ to $\overline{P}_0'$ while the point Q moves on $\mathfrak{E}_0$ from P_0' to E_0; at the same time the line $\tilde{\vartheta}_0$ revolves about P_0 from the position ϑ_0, in the sense of the arrow, into the position $\bar{\vartheta}_0$.

The conjugate points thus defined play for the discontinuous solutions a rôle similar to that of the ordinary conjugate points for continuous solutions, at least in the case when the line $\tilde{\vartheta}_0$ lies inside the angle of the two branches $P_1 P_0$, $P_0 P_2$. We refer in this respect to Caratheodory's dissertation, § 9.

THE UNIVERSITY OF CHICAGO, *January* 29, 1907.

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* Caratheodory restricts, however, the definition to the case when the line $\tilde{\vartheta}_0$ lies inside of the angle of the two branches $P_1 P_0$, $P_0 P_2$.

Mathematical Logic as based on the Theory of Types.

By Bertrand Russell.

The following theory of symbolic logic recommended itself to me in the first instance by its ability to solve certain contradictions, of which the one best known to mathematicians is Burali-Forti's concerning the greatest ordinal.* But the theory in question seems not wholly dependent on this indirect recommendation; it has also, if I am not mistaken, a certain consonance with common sense which makes it inherently credible. This, however, is not a merit upon which much stress should be laid; for common sense is far more fallible than it likes to believe. I shall therefore begin by stating some of the contradictions to be solved, and shall then show how the theory of logical types effects their solution.

I.

The Contradictions.

(1) The oldest contradiction of the kind in question is the *Epimenides.* Epimenides the Cretan said that all Cretans were liars, and all other statements made by Cretans were certainly lies. Was this a lie? The simplest form of this contradiction is afforded by the man who says "I am lying;" if he is lying, he is speaking the truth, and vice versa.

(2) Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, "x is a w" is equivalent† to "x is not an x." Hence, giving to x the value w, "w is a w" is equivalent to "w is not a w."

(3) Let T be the relation which subsists between two relations R and S whenever R does not have the relation R to S. Then, whatever relations R and S may be, "R has the relation T to S" is equivalent to "R does not have the

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*See below.

† Two propositions are called *equivalent* when both are true or both are false.

relation R to S." Hence, giving the value T to both R and S, "T has the relation T to T" is equivalent to "T does not have the relation T to T."

(4) The number of syllables in the English names of finite integers tends to increase as the integers grow larger, and must gradually increase indefinitely, since only a finite number of names can be made with a given finite number of syllables. Hence the names of some integers must consist of at least nineteen syllables, and among these there must be a least. Hence "the least integer not nameable in fewer than nineteen syllables" must denote a definite integer; in fact, it denotes 111,777. But "the least integer not nameable in fewer than nineteen syllables" is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction.*

(5) Among transfinite ordinals some can be defined, while others can not; for the total number of possible definitions is $\aleph_0$, while the number of transfinite ordinals exceeds $\aleph_0$. Hence there must be indefinable ordinals, and among these there must be a least. But this is defined as "the least indefinable ordinal," which is a contradiction.†

(6) Richard's paradox ‡ is akin to that of the least indefinable ordinal. It is as follows: Consider all decimals that can be defined by means of a finite number of words; let E be the class of such decimals. Then E has $\aleph_0$ terms; hence its members can be ordered as the 1st, 2nd, 3rd, ... Let N be a number defined as follows: If the nth figure in the nth decimal is p, let the nth figure in N be $p+1$ (or 0, if $p=9$). Then N is different from all the members of E, since, whatever finite value n may have, the nth figure in N is different from the nth figure in the nth of the decimals composing E, and therefore N is different from the nth decimal. Nevertheless we have defined N in a finite number of words, and therefore N ought to be a member of E. Thus N both is and is not a member of E.

(7) Burali-Forti's contradiction § may be stated as follows: It can be shown

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*This contradiction was suggested to me by Mr. G. G. Berry of the Bodleian Library.

† *Cf.* König, "Ueber die Grundlagen der Mengenlehre und das Kontinuumproblem," *Math. Annalen*, Vol. LXI (1905); A. C. Dixon, "On 'well-ordered' aggregates," *Proc. London Math. Soc.*, Series 2, Vol. IV, Part I (1906); and E. W. Hobson, "On the Arithmetic Continuum," *ibid.* The solution offered in the last of these papers does not seem to me adequate.

‡ *Cf.* Poincaré, "Les mathématiques et la logique," *Revue de Métaphysique et de Morale*, Mai, 1906, especially sections VII and IX; also Peano, *Revista de Mathematica*, Vol. VIII, No. 5 (1906), p. 149 ff.

§ "Una questione sui numeri transfiniti," *Rendiconti del circolo matematico di Palermo*, Vol. XI (1897).

that every well-ordered series has an ordinal number, that the series of ordinals up to and including any given ordinal exceeds the given ordinal by one, and (on certain very natural assumptions) that the series of all ordinals (in order of magnitude) is well-ordered. It follows that the series of all ordinals has an ordinal number, Ω say. But in that case the series of all ordinals including Ω has the ordinal number $\Omega + 1$, which must be greater than Ω. Hence Ω is not the ordinal number of all ordinals.

In all the above contradictions (which are merely selections from an indefinite number) there is a common characteristic, which we may describe as self-reference or reflexiveness. The remark of Epimenides must include itself in its own scope If *all* classes, provided they are not members of themselves, are members of w, this must also apply to w; and similarly for the analogous relational contradiction. In the cases of names and definitions, the paradoxes result from considering non-nameability and indefinability as elements in names and definitions. In the case of Burali-Forti's paradox, the series whose ordinal number causes the difficulty is the series of all ordinal numbers. In each contradiction something is said about *all* cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which *all* were concerned in what was said. Let us go through the contradictions one by one and see how this occurs.

(1) When a man says "I am lying," we may interpret his statement as: "There is a proposition which I am affirming and which is false." All statements that "there is" so-and-so may be regarded as denying that the opposite is always true; thus "I am lying" becomes: "It is not true of all propositions that either I am not affirming them or they are true;" in other words, "It is not true for all propositions p that if I affirm p, p is true." The paradox results from regarding this statement as affirming a proposition, which must therefore come within the scope of the statement. This, however, makes it evident that the notion of "all propositions" is illegitimate; for otherwise, there must be propositions (such as the above) which are about all propositions, and yet can not, without contradiction, be included among the propositions they are about. Whatever we suppose to be the totality of propositions, statements about this totality generate new propositions which, on pain of contradiction, must lie outside the totality. It is useless to enlarge the totality, for that equally enlarges the scope of statements about the totality. Hence there must be no totality of propositions, and "all propositions" must be a meaningless phrase.

(2) In this case, the class w is defined by reference to "all classes," and then turns out to be one among classes. If we seek help by deciding that no class is a member of itself, then w becomes the class of all classes, and we have to decide that this is not a member of itself, *i. e.*, is not a class. This is only possible if there is no such thing as the class of all classes in the sense required by the paradox. That there is no such class results from the fact that, if we suppose there is, the supposition immediately gives rise (as in the above contradiction) to new classes lying outside the supposed total of all classes.

(3) This case is exactly analogous to (2), and shows that we can not legitimately speak of "all relations."

(4) "The least integer not nameable in fewer than nineteen syllables" involves the totality of names, for it is "the least integer such that all names either do not apply to it or have more than nineteen syllables." Here we assume, in obtaining the contradiction, that a phrase containing "all names" is itself a name, though it appears from the contradiction that it can not be one of the names which were supposed to be all the names there are. Hence "all names" is an illegitimate notion.

(5) This case, similarly, shows that "all definitions" is an illegitimate notion.

(6) This is solved, like (5), by remarking that "all definitions" is an illegitimate notion. Thus the number E is *not* defined in a finite number of words, being in fact not defined at all.*

(7) Burali-Forti's contradiction shows that "all ordinals" is an illegitimate notion; for if not, all ordinals in order of magnitude form a well-ordered series, which must have an ordinal number greater than all ordinals.

Thus all our contradictions have in common the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself.

This leads us to the rule: "Whatever involves *all* of a collection must not be one of the collection;" or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total."†

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* *Cf.* "Les paradoxes de la logique," by the present author, *Revue de Métaphysique et de Morale*, Sept., 1906, p. 645.

† When I say that a collection has no total, I mean that statements about *all* its members are nonsense. Furthermore, it will be found that the use of this principle requires the distinction of *all* and *any* considered in Section II.

The above principle is, however, purely negative in its scope. It suffices to show that many theories are wrong, but it does not show how the errors are to be rectified. We can not say: "When I speak of *all* propositions, I mean all except those in which 'all propositions' are mentioned;" for in this explanation we have mentioned the propositions in which all propositions are mentioned, which we can not do significantly. It is impossible to avoid mentioning a thing by mentioning that we won't mention it. One might as well, in talking to a man with a long nose, say: "When I speak of noses, I except such as are inordinately long," which would not be a very successful effort to avoid a painful topic. Thus it is necessary, if we are not to sin against the above negative principle, to construct our logic without mentioning such things as "all propositions" or "all properties," and without even having to say that we are excluding such things. The exclusion must result naturally and inevitably from our positive doctrines, which must make it plain that "all propositions" and "all properties" are meaningless phrases.

The first difficulty that confronts us is as to the fundamental principles of logic known under the quaint name of "laws of thought." "All propositions are either true or false," for example, has become meaningless. If it were significant, it would be a proposition, and would come under its own scope. Nevertheless, some substitute must be found, or all general accounts of deduction become impossible.

Another more special difficulty is illustrated by the particular case of mathematical induction. We want to be able to say: "If n is a finite integer, n has all properties possessed by 0 and by the successors of all numbers possessing them." But here "all properties" must be replaced by some other phrase not open to the same objections. It might be thought that "all properties possessed by 0 and by the successors of all numbers possessing them" might be legitimate even if "all properties" were not. But in fact this is not so. We shall find that phrases of the form "all properties which *etc.*" involve *all* properties of which the "*etc.*" can be significantly either affirmed or denied, and not only those which in fact have whatever characteristic is in question; for, in the absence of a catalogue of properties having this characteristic, a statement about all those that have the characteristic must be hypothetical, and of the form: "It is always true that, if a property has the said characteristic, then etc." Thus mathematical induction is *primâ facie* incapable of being significantly enunciated, if "all properties" is a phrase destitute of meaning. This difficulty, as we shall see later, can be avoided; for the present we must consider the laws of logic, since these are far more fundamental.

II.

All and Any.

Given a statement containing a variable x, say "$x = x$," we may affirm that this holds in all instances, or we may affirm any one of the instances without deciding as to which instance we are affirming. The distinction is roughly the same as that between the general and particular enunciation in Euclid. The general enunciation tells us something about (say) all triangles, while the particular enunciation takes one triangle, and asserts the same thing of this one triangle. But the triangle taken is *any* triangle, not some one special triangle; and thus although, throughout the proof, only one triangle is dealt with, yet the proof retains its generality. If we say: "Let ABC be a triangle, then the sides AB, AC are together greater than the side BC," we are saying something about *one* triangle, not about *all* triangles; but the one triangle concerned is absolutely ambiguous, and our statement consequently is also absolutely ambiguous. We do not affirm any one definite proposition, but an undetermined one of all the propositions resulting from supposing ABC to be this or that triangle. This notion of ambiguous assertion is very important, and it is vital not to confound an ambiguous assertion with the definite assertion that the same thing holds in *all* cases.

The distinction between (1) asserting any value of a propositional function, and (2) asserting that the function is always true, is present throughout mathematics, as it is in Euclid's distinction of general and particular enunciations. In any chain of mathematical reasoning, the objects whose properties are being investigated are the arguments to *any* value of some propositional function. Take as an illustration the following definition:

"We call $f(x)$ continuous for $x = a$ if, for every positive number σ, different from 0, there exists a positive number ε, different from 0, such that, for all values of δ which are numerically less than ε, the difference $f(a+\delta) - f(a)$ is numerically less than σ."

Here the function f is *any* function for which the above statement has a meaning; the statement is *about* f, and varies as f varies. But the statement is not *about* σ or ε or δ, because *all* possible values of these are concerned, not one undetermined value. (In regard to ε, the statement "there exists a positive number ε such that *etc.*" is the denial that the denial of "*etc.*" is true of *all* positive numbers.) For this reason, when *any* value of a propositional function is asserted, the argument (*e.g.*, f in the above) is called a *real* variable; whereas, when a function is said to be *always* true, or to be not always true, the argument is called an *apparent* variable.* Thus in the above definition, f is a real variable, and σ, ε, δ are apparent variables.

When we assert *any* value of a propositional function, we shall say simply that we assert the *propositional function.* Thus if we enunciate the law of identity in the form "$x = x$," we are asserting the function "$x = x$;" *i. e.*, we are asserting any value of this function. Similarly we may be said to deny a propositional function when we deny any instance of it. We can only truly assert a propositional function if, whatever value we choose, that value is true; similarly we can only truly deny it if, whatever value we choose, that value is false. Hence in the general case, in which some values are true and some false, we can neither assert nor deny a propositional function.†

If ϕx is a propositional function, we will denote by "$(x) . \phi x$" the proposition "ϕx is always true." Similarly "$(x, y) . \phi(x, y)$" will mean "$\phi(x, y)$ is always true," and so on. Then the distinction between the assertion of all values and the assertion of any is the distinction between (1) asserting $(x) . \phi x$ and (2) asserting ϕx where x is undetermined. The latter differs from the former in that it can not be treated as one determinate proposition.

The distinction between asserting ϕx and asserting $(x) . \phi x$ was, I believe, first emphasized by Frege.‡ His reason for introducing the distinction explicitly was the same which had caused it to be present in the practice of mathematicians; namely, that deduction can only be effected with *real* variables, not with apparent variables. In the case of Euclid's proofs, this is evident: we need (say) some one triangle ABC to reason about, though it does not matter what triangle it is. The triangle ABC is a *real* variable; and although it is *any* triangle, it remains the *same* triangle throughout the argument. But in the general enunciation,

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*These two terms are due to Peano, who uses them approximately in the above sense. *Cf.*, *e. g.*, *Formulaire Mathématique*, Vol. IV, p. 5 (Turin, 1903).

†Mr. MacColl speaks of "propositions" as divided into the three classes of certain, variable, and impossible. We may accept this division as applying to propositional functions. A function which can be asserted is certain, one which can be denied is impossible, and all others are (in Mr. MacColl's sense) variable.

‡See his *Grundgesetze der Arithmetik*, Vol. I (Jena, 1893), §17, p. 31.

the triangle is an apparent variable. If we adhere to the apparent variable, we can not perform any deductions, and this is why in all proofs, real variables have to be used. Suppose, to take the simplest case, that we know "ϕx is always true," *i. e.* "$(x) . \phi x$," and we know "ϕx always implies ψx," *i. e.* "$(x) . \{\phi x$ implies $\psi x\}$." How shall we infer "ψx is always true," *i. e.* "$(x) . \psi x$?" We know it is always true that if ϕx is true, and if ϕx implies ψx, then ψx is true. But we have no premises to the effect that ϕx is true and ϕx implies ψx; what we have is: ϕx is *always* true, and ϕx *always* implies ψx. In order to make our inference, we must go from "ϕx is always true" to ϕx, and from "ϕx always implies ψx" to "ϕx implies ψx," where the x, while remaining any possible argument, is to be the same in both. Then, from "ϕx" and "ϕx implies ψx," we infer "ψx;" thus ψx is true for any possible argument, and therefore is always true. Thus in order to infer "$(x) . \psi x$ from "$(x) . \phi x$" and "$(x) . \{\phi x$ implies $\psi x\}$," we have to pass from the apparent to the real variable, and then back again to the apparent variable. This process is required in all mathematical reasoning which proceeds from the assertion of all values of one or more propositional functions to the assertion of all values of some other propositional function, as, *e. g.*, from "all isosceles triangles have equal angles at the base" to "all triangles having equal angles at the base are isosceles." In particular, this process is required in proving *Barbara* and the other moods of the syllogism. In a word, *all deduction operates with real variables* (or with constants).

It might be supposed that we could dispense with apparent variables altogether, contenting ourselves with *any* as a substitute for *all*. This, however, is not the case. Take, for example, the definition of a continuous function quoted above: in this definition σ, ε, and δ must be apparent variables. Apparent variables are constantly required for definitions. Take, *e. g.*, the following: "An integer is called a *prime* when it has no integral factors except 1 and itself." This definition unavoidably involves an apparent variable in the form: "If n is an integer other than 1 or the given integer, n is not a factor of the given integer, for all possible values of n"

The distinction between *all* and *any* is, therefore, necessary to deductive reasoning, and occurs throughout mathematics; though, so far as I know, its importance remained unnoticed until Frege pointed it out.

For our purposes it has a different utility, which is very great. In the case of such variables as propositions or properties, "any value" is legitimate, though "all values" is not. Thus we may say: "p is true or false, where p is any proposition," though we can not say "all propositions are true or false." The reason is that, in the former, we merely affirm an undetermined one of the propositions of the form "p is true or false," whereas in the latter we affirm (if anything) a new proposition, different from all the propositions of the form "p is true or false." Thus we may admit "any value" of a variable in cases where "all values" would lead to reflexive fallacies; for the admission of "any value" does not in the same way create new values. Hence the fundamental laws of logic can be stated concerning *any* proposition, though we can not significantly say that they hold of *all* propositions. These laws have, so to speak, a particular enunciation but no general enunciation. There is no one proposition which *is* the law of contradiction (say); there are only the various instances of the law. Of any proposition p, we can say: "p and not-p can not both be true;" but there is no such proposition as: "Every proposition p is such that p and not-p can not both be true."

A similar explanation applies to properties. We can speak of *any* property of x, but not of *all* properties, because new properties would be thereby generated. Thus we can say: "If n is a finite integer, and if 0 has the property ϕ, and $m+1$ has the property ϕ provided m has it, it follows that n has the property ϕ." Here we need not specify ϕ; ϕ stands for "any property." But we can not say: "A finite integer is defined as one which has *every* property ϕ possessed by 0 and by the successors of possessors." For here it is essential to consider *every* property,* not *any* property; and in using such a definition we assume that it embodies a *property* distinctive of finite integers, which is just the kind of assumption from which, as we saw, the reflexive contradictions spring.

In the above instance, it is necessary to avoid the suggestions of ordinary language, which is not suitable for expressing the distinction required. The point may be illustrated further as follows: If induction is to be used for defining finite integers, induction must state a definite property of finite integers, not an ambiguous property. But if ϕ is a real variable, the statement "n has the property ϕ provided this property is possessed by 0 and by the successors of possessors" assigns to n a property which varies as ϕ varies, and such a property can not be used to define the class of finite integers. We wish to say: "'n is a finite integer' means: 'Whatever property ϕ may be, n has the property ϕ provided ϕ is possessed by 0 and by the successors of possessors.'" But here ϕ has become an *apparent* variable. To keep it a real variable, we should have to say: "Whatever property ϕ may be, 'n is a finite integer' means: 'n has the property ϕ provided ϕ is possessed by 0 and by the successors of possessors.'" But here the meaning of 'n is a finite integer' varies as ϕ varies, and thus such a definition is impossible. This case illustrates an important point, namely the following: "The scope* of a real variable can never be less than the whole propositional function in the assertion of which the said variable occurs." That is, if our propositional function is (say) "ϕx implies p," the assertion of this function will mean "any value of 'ϕx implies p' is true," *not* "'any value of ϕx is true' implies p." In the latter, we have really "*all* values of ϕx are true," and the x is an *apparent* variable.

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* This is indistinguishable from "all properties."

III.

The Meaning and Range of Generalized Propositions.

In this section we have to consider first the meaning of propositions in which the word *all* occurs, and then the kind of collections which admit of propositions about all their members.

It is convenient to give the name *generalized propositions* not only to such as contain *all*, but also to such as contain *some* (undefined). The proposition "ϕx is sometimes true" is equivalent to the denial of "not-ϕx is always true;" "some A is B" is equivalent to the denial of "all A is not B;" *i. e.*, of "no A is B." Whether it is possible to find interpretations which distinguish "ϕx is sometimes true" from the denial of "not-ϕx is always true," it is unnecessary to inquire; for our purposes we may *define* "ϕx is sometimes true" as the denial of "not-ϕx is always true." In any case, the two kinds of propositions require the same kind of interpretation, and are subject to the same limitations. In each there is an apparent variable; and it is the presence of an apparent variable which constitutes what I mean by a generalized proposition. (Note that there can not be a *real* variable in any proposition; for what contains a real variable is a propositional function, not a proposition.)

The first question we have to ask in this section is: How are we to interpret the word *all* in such propositions as "all men are mortal?" At first sight, it might be thought that there could be no difficulty, that "all men" is a perfectly clear idea, and that we say of all men that they are mortal. But to this view there are many objections.

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* The *scope* of a real variable is the whole function of which "any value" is in question. Thus in "ϕx implies p" the scope of x is not ϕx, but "ϕx implies p."

(1) If this view were right, it would seem that "all men are mortal" could not be true if there were no men. Yet, as Mr. Bradley has urged,* "Trespassers will be prosecuted" may be perfectly true even if no one trespasses; and hence, as he further argues, we are driven to interpret such propositions as hypotheticals, meaning "if anyone trespasses, he will be prosecuted;" *i. e.*, "if x trespasses, x will be prosecuted," where the range of values which x may have, whatever it is, is certainly not confined to those who really trespass. Similarly "all men are mortal" will mean "if x is a man, x is mortal, where x may have any value within a certain range." What this range is, remains to be determined; but in any case it is wider than "men," for the above hypothetical is certainly often true when x is not a man.

(2) "All men" is a denoting phrase; and it would appear, for reasons which I have set forth elsewhere,† that denoting phrases never have any meaning in isolation, but only enter as constituents into the verbal expression of propositions which contain no constituent corresponding to the denoting phrases in question. That is to say, a denoting phrase is defined by means of the propositions in whose verbal expression it occurs. Hence it is impossible that these propositions should acquire their meaning through the denoting phrases; we must find an independent interpretation of the propositions containing such phrases, and must not use these phrases in explaining what such propositions mean. Hence we can not regard "all men are mortal" as a statement about "all men."

(3) Even if there were such an object as "all men," it is plain that it is not this object to which we attribute mortality when we say "all men are mortal." If we were attributing mortality to this object, we should have to say "*all men* is mortal." Thus the supposition that there is such an object as "all men" will not help us to interpret "all men are mortal."

(4) It seems obvious that, if we meet something which may be a man or may be an angel in disguise, it comes within the scope of "all men are mortal" to assert "if this is a man, it is mortal." Thus again, as in the case of the trespassers, it seems plain that we are really saying "if anything is a man, it is mortal," and that the question whether this or that is a man does not fall within the scope of our assertion, as it would do if the *all* really referred to "all men."

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* *Logic*, Part I, Chapter II.

† "On Denoting," *Mind*, October, 1905.

(5) We thus arrive at the view that what is meant by "all men are mortal" may be more explicitly stated in some such form as "it is always true that if x is a man, x is mortal." Here we have to inquire as to the scope of the word *always*.

(6) It is obvious that *always* includes some cases in which x is not a man, as we saw in the case of the disguised angel. If x were limited to the case when x is a man, we could infer that x is a mortal, since if x is a man, x is a mortal. Hence, with the same meaning of *always*, we should find "it is always true that x is mortal." But it is plain that, without altering the meaning of *always*, this new proposition is false, though the other was true.

(7) One might hope that "always" would mean "for all values of x." But "all values of x," if legitimate, would include as parts "all propositions" and "all functions," and such illegitimate totalities. Hence the values of x must be somehow restricted within some legitimate totality. This seems to lead us to the traditional doctrine of a "universe of discourse" within which x must be supposed to lie.

(8) Yet it is quite essential that we should have some meaning of *always* which does not have to be expressed in a restrictive hypothesis as to x. For suppose "always" means "whenever x belongs to the class i." Then "all men are mortal" becomes "whenever x belongs to the class i, if x is a man, x is mortal;" *i. e.*, "it is always true that if x belongs to the class i, then, if x is a man, x is mortal." But what is our new *always* to mean? There seems no more reason for restricting x, in this new proposition, to the class i, than there was before for restricting it to the class *man*. Thus we shall be led on to a new wider universe, and so on *ad infinitum*, unless we can discover some natural restriction upon the possible values of (*i. e.*, some restriction given with) the function "if x is a man, x is mortal," and not needing to be imposed from without.

(9) It seems obvious that, since all men are mortal, there can not be any *false* proposition which is a value of the function "if x is a man, x is mortal." For if this is a proposition at all, the hypothesis "x is a man" must be a proposition, and so must the conclusion "x is mortal." But if the hypothesis is false, the hypothetical is true; and if the hypothesis is true, the hypothetical is true. Hence there can be no false propositions of the form "if x is a man, x is mortal."

(10) It follows that, if any values of x are to be excluded, they can only be values for which there is no proposition of the form "if x is a man, x is mortal;"

i. e., for which this phrase is meaningless. Since, as we saw in (7), there must be excluded values of x, it follows that the function "if x is a man, x is mortal" must have a certain *range of significance*,[*] which falls short of all imaginable values of x, though it exceeds the values which are men. The restriction on x is therefore a restriction to the range of significance of the function "if x is a man, x is mortal."

(11) We thus reach the conclusion that "all men are mortal" means "if x is a man, x is mortal, always," where *always* means "for all values of the function 'if x is a man, x is mortal.'" This is an *internal* limitation upon x, given by the nature of the function; and it is a limitation which does not require explicit statement, since it is impossible for a function to be true more generally than for all its values. Moreover, if the range of significance of the function is i, the function "if x is an i, then if x is a man, x is mortal" has the same range of significance, since it can not be significant unless its constituent "if x is a man, x is mortal" is significant. But here the range of significance is again implicit, as it was in 'if x is a man, x is mortal;' thus we can not make ranges of significance explicit, since the attempt to do so only gives rise to a new proposition in which the same range of significance is implicit.

Thus generally: "$(x) . \phi x$" is to mean "ϕx always." This may be interpreted, though with less exactitude, as "ϕx is always true," or, more explicitly: "All propositions of the form ϕx are true," or "All values of the function ϕx are true."[†] Thus the fundamental *all* is "all values of a propositional function," and every other *all* is derivative from this. And every propositional function has a certain *range of significance*, within which lie the arguments for which the function has values. Within this range of arguments, the function is true or false; outside this range, it is nonsense.

The above argumentation may be summed up as follows:

The difficulty which besets attempts to restrict the variable is, that restrictions naturally express themselves as hypotheses that the variable is of such or such a kind, and that, when so expressed, the resulting hypothetical is free from the intended restriction. For example, let us attempt to restrict the variable to *men*, and assert that, subject to this restriction, "x is mortal" is always true. Then what is always true is that if x is a man, x is mortal; and this hypothetical is true even when x is not a man. Thus a variable can never be restricted within a certain range if the propositional function in which the variable occurs remains significant when the variable is outside that range. But if the function ceases to be significant when the variable goes outside a certain range, then the variable is *ipso facto* confined to that range, without the need of any explicit statement to that effect. This principle is to be borne in mind in the development of logical types, to which we shall shortly proceed.

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[*] A function is said to be significant for the argument x if it has a value for this argument. Thus we may say shortly "ϕx is significant," meaning "the function ϕ has a value for the argument x." The range of significance of a function consists of all the arguments for which the function is true, together with all the arguments for which it is false.

[†] A linguistically convenient expression for this idea is: "ϕx is true for all *possible* values of x," a possible value being understood to be one for which ϕx is significant.

We can now begin to see how it comes that "all so-and-so's" is sometimes a legitimate phrase and sometimes not. Suppose we say "all terms which have the property ϕ have the property ψ." That means, according to the above interpretation, "ϕx always implies ψx." Provided the range of significance of ϕx is the same as that of ψx, this statement is significant; thus, given any definite function ϕx, there are propositions about "all the terms satisfying ϕx." But it sometimes happens (as we shall see more fully later on) that what appears verbally as one function is really many analogous functions with different ranges of significance. This applies, for example, to "p is true," which, we shall find, is not really one function of p, but is different functions according to the kind of proposition that p is. In such a case, the *phrase* expressing the ambiguous function may, owing to the ambiguity, be significant throughout a set of values of the argument exceeding the range of significance of any one function. In such a case, *all* is not legitimate. Thus if we try to say "all true propositions have the property ϕ," *i. e.*, "'p is true' always implies ϕp," the possible arguments to 'p is true' necessarily exceed the possible arguments to ϕ, and therefore the attempted general statement is impossible. For this reason, genuine general statements about all true propositions can not be made. It may happen, however, that the supposed function ϕ is really ambiguous like 'p is true;' and if it happens to have an ambiguity precisely of the same kind as that of 'p is true,' we may be able always to give an interpretation to the proposition "'p is true' implies ϕp." This will occur, *e. g.*, if ϕp is "not-p is false." Thus we get an appearance, in such cases, of a general proposition concerning *all* propositions; but this appearance is due to a systematic ambiguity about such words as *true* and *false*. (This systematic ambiguity results from the hierarchy of propositions which will be explained later on). We may, in all such cases, make our statement about *any* proposition, since the meaning of the ambiguous words will adapt itself to any proposition. But if we turn our proposition into an apparent variable, and say something about *all*, we must suppose the ambiguous words fixed to this or that possible meaning, though it may be quite irrelevant which of their possible meanings they are to have. This is how it happens both that *all* has limitations which exclude "all propositions," and that there nevertheless *seem* to be true statements about "all propositions." Both these points will become plainer when the theory of types has been explained.

It has often been suggested* that what is required in order that it may be legitimate to speak of *all* of a collection is that the collection should be finite. Thus "all men are mortal" will be legitimate because men form a finite class. But that is not really the reason why we can speak of "all men." What is essential, as appears from the above discussion, is not finitude, but what may be called *logical homogeneity.* This property is to belong to any collection whose terms are all contained within the range of significance of some one function. It would always be obvious at a glance whether a collection possessed this property or not, if it were not for the concealed ambiguity in common logical terms such as *true* and *false*, which gives an appearance of being a single function to what is really a conglomeration of many functions with different ranges of significance.

The conclusions of this section are as follows: Every proposition containing *all* asserts that some propositional function is always true; and this means that all values of the said function are true, not that the function is true for all arguments, since there are arguments for which any given function is meaningless, *i. e.*, has no value. Hence we can speak of *all* of a collection when and only when the collection forms part or the whole of the *range of significance* of some propositional function, the range of significance being defined as the collection of those arguments for which the function in question is significant, *i. e.*, has a value.

IV.

The Hierarchy of Types.

A *type* is defined as the range of significance of a propositional function, *i. e.*, as the collection of arguments for which the said function has values. Whenever an apparent variable occurs in a proposition, the range of values of the apparent variable is a type, the type being fixed by the function of which "all

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**E. g.*, by M. Poincaré, *Revue de Métaphysique et de Morale*, Mai, 1906.

values" are concerned. The division of objects into types is necessitated by the reflexive fallacies which otherwise arise. These fallacies, as we saw, are to be avoided by what may be called the "vicious-circle principle;" *i. e.*, "no totality can contain members defined in terms of itself." This principle, in our technical language, becomes: "Whatever contains an apparent variable must not be a possible value of that variable." Thus whatever contains an apparent variable must be of a different type from the possible values of that variable; we will say that it is of a *higher* type. Thus the apparent variables contained in an expression are what determines its type. This is the guiding principle in what follows.

Propositions which contain apparent variables are generated from such as do not contain these apparent variables by processes of which one is always the process of *generalization*, *i. e.*, the substitution of a variable for one of the terms of a proposition, and the assertion of the resulting function for all possible values of the variable. Hence a proposition is called a *generalized* proposition when it contains an apparent variable. A proposition containing no apparent variable we will call an *elementary* proposition. It is plain that a proposition containing an apparent variable presupposes others from which it can be obtained by generalization; hence all generalized propositions presuppose elementary propositions. In an elementary proposition we can distinguish one or more *terms* from one or more *concepts;* the *terms* are whatever can be regarded as the *subject* of the proposition, while the concepts are the predicates or relations asserted of these terms.* The terms of elementary propositions we will call *individuals;* these form the first or lowest type.

It is unnecessary, in practice, to know what objects belong to the lowest type, or even whether the lowest type of variable occurring in a given context is that of individuals or some other. For in practice only the *relative* types of variables are relevant; thus the lowest type occurring in a given context may be called that of individuals, so far as that context is concerned. It follows that the above account of individuals is not essential to the truth of what follows; all that is essential is the way in which other types are generated from individuals, however the type of individuals may be constituted.

By applying the process of generalization to individuals occurring in elementary propositions, we obtain new propositions. The legitimacy of this

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*See *Principles of Mathematics*, §48.

process requires only that no individuals should be propositions. That this is so, is to be secured by the meaning we give to the word *individual.* We may define an individual as something destitute of complexity; it is then obviously not a proposition, since propositions are essentially complex. Hence in applying the process of generalization to individuals we run no risk of incurring reflexive fallacies.

Elementary propositions together with such as contain only individuals as apparent variables we will call *first-order propositions.* These form the second logical type.

We have thus a new totality, that of *first-order propositions.* We can thus form new propositions in which first-order propositions occur as apparent variables. These we will call *second-order propositions;* these form the third logical type. Thus, *e. g.*, if Epimenides asserts "all first-order propositions affirmed by me are false," he asserts a second-order proposition; he may assert this truly, without asserting truly any first-order proposition, and thus no contradiction arises.

The above process can be continued indefinitely. The $n+1$th logical type will consist of propositions of order n, which will be such as contain propositions of order $n-1$, but of no higher order, as apparent variables. The types so obtained are mutually exclusive, and thus no reflexive fallacies are possible so long as we remember that an apparent variable must always be confined within some one type.

In practice, a hierarchy of *functions* is more convenient than one of propositions. Functions of various orders may be obtained from propositions of various orders by the method of *substitution.* If p is a proposition, and a a constituent of p, let "$p/a`x$" denote the proposition which results from substituting $x$ for $a$ wherever $a$ occurs in $p$. Then $p/a$, which we will call a *matrix*, may take the place of a function; its value for the argument $x$ is $p/a`x$, and its value for the argument a is p. Similarly, if "$p/(a, b)`(x, y)$" denotes the result of first substituting x for a and then substituting y for b, we may use the double matrix $p/(a, b)$ to represent a double function. In this way we can avoid apparent variables other than individuals and propositions of various orders. The *order* of a matrix will be defined as being the order of the proposition in which the substitution is effected, which proposition we will call the *prototype.* The order of a matrix does not determine its type: in the first place because it does not determine the number of arguments for which others are to be substituted (*i. e.*, whether the matrix is of the form p/a or $p/(a, b)$ or $p/(a, b, c)$ etc.); in the second place because, if the prototype is of more than the first order, the arguments may be either propositions or individuals. But it is plain that the type of a matrix is definable always by means of the hierarchy of propositions.

Although it is *possible* to replace functions by matrices, and although this procedure introduces a certain simplicity into the explanation of types, it is technically inconvenient. Technically, it is convenient to replace the prototype p by ϕa, and to replace $p/a \text{'} x$ by ϕx; thus where, if matrices were being employed, p and a would appear as apparent variables, we now have ϕ as our apparent variable. In order that ϕ may be legitimate as an apparent variable, it is necessary that its values should be confined to propositions of some one type. Hence we proceed as follows.

A function whose argument is an individual and whose value is always a first-order proposition will be called a first-order function. A function involving a first-order function or proposition as apparent variable will be called a second-order function, and so on. A function of one variable which is of the order next above that of its argument will be called a *predicative* function; the same name will be given to a function of several variables if there is one among these variables in respect of which the function becomes predicative when values are assigned to all the other variables. Then the type of a function is determined by the type of its values and the number and type of its arguments.

The hierarchy of functions may be further explained as follows. A first-order function of an individual x will be denoted by $\phi \,!\, x$ (the letters ψ, χ, θ, f, g, F, G will also be used for functions). No first-order function contains a function as apparent variable; hence such functions form a well-defined totality, and the ϕ in $\phi \,!\, x$ can be turned into an apparent variable. Any proposition in which ϕ appears as apparent variable, and there is no apparent variable of higher type than ϕ, is a second-order proposition. If such a proposition contains an individual x, it is not a predicative function of x; but if it contains a first-order function ϕ, it is a predicative function of ϕ, and will be written $f \,!\, (\psi \,!\, \hat{z})$. Then f is a *second-order predicative function;* the possible values of f again form a well-defined totality, and we can turn f into an apparent variable. We can thus define *third-order predicative functions,* which will be such as have third-order propositions for their values and second-order predicative functions for their arguments. And in this way we can proceed indefinitely. A precisely similar development applies to functions of several variables.

We will adopt the following conventions. Variables of the lowest type occurring in any context will be denoted by small Latin letters (excluding f and g, which are reserved for functions); a predicative function of an argument x (where x may be of any type) will be denoted by $\phi ! x$ (where ψ, χ, θ, f, g, F or G may replace ϕ); similarly a predicative function of two arguments x and y will be denoted by $\phi ! (x, y)$; a general function of x will be denoted by ϕx, and a general function of x and y by $\phi(x, y)$. In ϕx, ϕ can not be made into an apparent variable, since its type is indeterminate; but in $\phi ! x$, where ϕ is a *predicative* function whose argument is of some given type, ϕ *can* be made into an apparent variable.

It is important to observe that since there are various types of propositions and functions, and since generalization can only be applied within some one type, all phrases containing the words "all propositions" or "all functions" are *primâ facie* meaningless, though in certain cases they are capable of an unobjectionable interpretation. The contradictions arise from the use of such phrases in cases where no innocent meaning can be found.

If we now revert to the contradictions, we see at once that some of them are solved by the theory of types. Wherever "all propositions" are mentioned, we must substitute "all propositions of order n," where it is indifferent what value we give to n, but it is essential that n should have *some* value. Thus when a man says "I am lying," we must interpret him as meaning: "There is a proposition of order n, which I affirm, and which is false." This is a proposition of order $n + 1$; hence the man is not affirming any proposition of order n; hence his statement is false, and yet its falsehood does not imply, as that of "I am lying" appeared to do, that he is making a true statement. This solves the liar.

Consider next "the least integer not nameable in fewer than nineteen syllables." It is to be observed, in the first place, that *nameable* must mean "nameable by means of such-and-such assigned names," and that the number of assigned names must be finite. For if it is not finite, there is no reason why there should be any integer not nameable in fewer than nineteen syllables, and the paradox collapses. We may next suppose that "nameable in terms of names of the class N" means "being the only term satisfying some function composed wholly of names of the class N." The solution of this paradox lies, I think, in the simple observation that "nameable in terms of names of the class N" is never itself nameable in terms of names of that class. If we enlarge N by adding the name "nameable in terms of names of the class N," our fundamental apparatus of names is enlarged; calling the new apparatus N', "nameable in terms of names of the class N'" remains not nameable in terms of names of the class N'. If we try to enlarge N till it embraces *all* names, "nameable" becomes (by what was said above) "being the only term satisfying some function composed wholly of names." But here there is a function as apparent variable; hence we are confined to predicative functions of some one type (for non-predicative functions can not be apparent variables). Hence we have only to observe that nameability in terms of such functions is non-predicative in order to escape the paradox.

The case of "the least indefinable ordinal" is closely analogous to the case we have just discussed. Here, as before, "definable" must be relative to some given apparatus of fundamental ideas; and there is reason to suppose that "definable in terms of ideas of the class N" is not definable in terms of ideas of the class N. It will be true that there is some definite segment of the series of ordinals consisting wholly of definable ordinals, and having the least indefinable ordinal as its limit. This least indefinable ordinal will be definable by a slight enlargement of our fundamental apparatus; but there will then be a new ordinal which will be the least that is indefinable with the new apparatus. If we enlarge our apparatus so as to include all possible ideas, there is no longer any reason to believe that there is any indefinable ordinal. The apparent force of the paradox lies largely, I think, in the supposition that if all the ordinals of a certain class are definable, the class must be definable, in which case its successor is of course also definable; but there is no reason for accepting this supposition.

The other contradictions, that of Burali-Forti in particular, require some further developments for their solution.

V.

The Axiom of Reducibility.

A propositional function of x may, as we have seen, be of any order; hence any statement about "all properties of x" is meaningless. (A "property of x" is the same thing as a "propositional function which holds of x.") But it is absolutely necessary, if mathematics is to be possible, that we should have some method of making statements which will usually be equivalent to what we have in mind when we (inaccurately) speak of "all properties of x." This necessity appears in many cases, but especially in connection with mathematical induction. We can say, by the use of *any* instead of *all*, "Any property possessed by 0, and by the successors of all numbers possessing it, is possessed by all finite numbers." But we can not go on to: "A finite number is one which possesses *all* properties possessed by 0 and by the successors of all numbers possessing them." If we confine this statement to all first-order properties of numbers, we can not infer that it holds of second-order properties. For example, we shall be unable to prove that if m, n are finite numbers, then $m + n$ is a finite number. For, with the above definition, "m is a finite number" is a second-order property of m; hence the fact that $m + 0$ is a finite number, and that, if $m + n$ is a finite number, so is $m + n + 1$, does not allow us to conclude by induction that $m + n$ is a finite number. It is obvious that such a state of things renders much of elementary mathematics impossible.

The other definition of finitude, by the non-similarity of whole and part, fares no better. For this definition is: "A class is said to be finite when every one-one relation whose domain is the class and whose converse domain is contained in the class has the whole class for its converse domain." Here a variable relation appears, *i. e.*, a variable function of two variables; we have to take *all* values of this function, which requires that it should be of some assigned order; but any assigned order will not enable us to deduce many of the propositions of elementary mathematics.

Hence we must find, if possible, some method of reducing the order of a propositional function without affecting the truth or falsehood of its values. This seems to be what common-sense effects by the admission of *classes*. Given any propositional function ϕx, of whatever order, this is assumed to be equivalent, for all values of x, to a statement of the form "x belongs to the class α." Now this statement is of the first order, since it makes no allusion to "all functions of such-and-such a type." Indeed its only practical advantage over the original statement ϕx is that it is of the first order. There is no advantage in assuming that there really are such things as classes, and the contradiction about the classes which are not members of themselves shows that, if there are classes, they must be something radically different from individuals. I believe the chief purpose which classes serve, and the chief reason which makes them linguistically convenient, is that they provide a method of reducing the order of a propositional function. I shall, therefore, not assume anything of what may seem to be involved in the common-sense admission of classes, except this: that every propositional function is equivalent, for all its values, to some predicative function.

This assumption with regard to functions is to be made whatever may be the type of their arguments. Let ϕx be a function, of any order, of an argument x, which may itself be either an individual or a function of any order. If ϕ is of the order next above x, we write the function in the form $\phi ! x$; in such a case we will call ϕ a *predicative* function. Thus a predicative function of an individual is a first-order function; and for higher types of arguments, predicative functions take the place that first-order functions take in respect of individuals. We assume, then, that every function is equivalent, for all its values, to some predicative function of the same argument. This assumption seems to be the essence of the usual assumption of classes; at any rate, it retains as much of classes as we have any use for, and little enough to avoid the contradictions which a less grudging admission of classes is apt to entail. We will call this assumption the *axiom of classes*, or the *axiom of reducibility.*

We shall assume similarly that every function of two variables is equivalent, for all its values, to a predicative function of those variables, where a predicative function of two variables is one such that there is one of the variables in respect of which the function becomes predicative (in our previous sense) when a value is assigned to the other variable. This assumption is what seems to be meant by saying that any statement about two variables defines a relation between them. We will call this assumption the *axiom of relations* or the *axiom of reducibility.*

In dealing with relations between more than two terms, similar assumptions would be needed for three, four, ... variables. But these assumptions are not indispensable for our purpose, and are therefore not made in this paper.

By the help of the axiom of reducibility, statements about "all first-order functions of x" or "all predicative functions of α" yield most of the results which otherwise would require "all functions." The essential point is that such results are obtained in all cases where only the truth or falsehood of values of the functions concerned are relevant, as is invariably the case in mathematics. Thus mathematical induction, for example, need now only be stated for all predicative functions of numbers; it then follows from the axiom of classes that it holds of *any* function of whatever order. It might be thought that the paradoxes for the sake of which we invented the hierarchy of types would now reappear. But this is not the case, because, in such paradoxes, either something beyond the truth or falsehood of values of functions is relevant, or expressions occur which are unmeaning even after the introduction of the axiom of reducibility. For example, such a statement as "Epimenides asserts ψx" is not equivalent to "Epimenides asserts $\phi ! x$," even though ψx and $\phi ! x$ are equivalent. Thus "I am lying" remains unmeaning if we attempt to include *all* propositions among those which I may be falsely affirming, and is unaffected by the axiom of classes if we confine it to propositions of order n. The hierarchy of propositions and functions, therefore, remains relevant in just those cases in which there is a paradox to be avoided.

VI.

Primitive Ideas and Propositions of Symbolic Logic.

The primitive ideas required in symbolic logic appear to be the following seven:

(1) Any propositional function of a variable x or of several variables $x, y, z, \ldots$ This will be denoted by ϕx or $\phi(x, y, z, \ldots)$

(2) The negation of a proposition. If p is the proposition, its negation will be denoted by $\sim p$.

(3) The disjunction or logical sum of two propositions; *i. e.*, "this or that." If p, q are the two propositions, their disjunction will be denoted by $p \vee q$.*

(4) The truth of *any* value of a propositional function; *i. e.*, of ϕx where x is not specified.

(5) The truth of *all* values of a propositional function. This is denoted by $(x) . \phi x$ or $(x) : \phi x$ or whatever larger number of dots may be necessary to bracket off the proposition.† In $(x) . \phi x$, x is called an *apparent variable*, whereas when ϕx is asserted, where x is not specified, x is called a *real variable*.

(6) Any predicative function of an argument of any type; this will be represented by $\phi ! x$ or $\phi ! \alpha$ or $\phi ! R$, according to circumstances. A predicative function of x is one whose values are propositions of the type next above that of x, if x is an individual or a proposition, or that of values of x if x is a

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*In a previous article in this journal, I took implication as indefinable, instead of disjunction. The choice between the two is a matter of taste; I now choose disjunction, because it enables us to diminish the number of primitive propositions.

†The use of dots follows Peano's usage. It is fully explained by Mr. Whitehead, "On Cardinal Numbers," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXIV, and "On Mathematical Concepts of the Material World," *Phil. Trans. A.*, Vol. CCV, p. 472.

function. It may be described as one in which the apparent variables, if any, are all of the same type as x or of lower type; and a variable is of lower type than x if it can significantly occur as argument to x, or as argument to an argument to x, etc.

(7) Assertion; *i. e.*, the assertion that some proposition is true, or that any value of some propositional function is true. This is required to distinguish a proposition actually asserted from one merely considered, or from one adduced as hypothesis to some other. It will be indicated by the sign "$\vdash$" prefixed to what is asserted, with enough dots to bracket off what is asserted.*

Before proceeding to the primitive propositions, we need certain definitions. In the following definitions, as well as in the primitive propositions, the letters p, q, r are used to denote propositions.

$$p \supset q \,.\!=.\, \sim p \vee q \quad \text{Df.}$$

This definition states that "$p \supset q$" (which is read "p implies q") is to mean "p is false or q is true." I do not mean to affirm that "implies" can not have any other meaning, but only that this meaning is the one which it is most convenient to give to "implies" in symbolic logic. In a definition, the sign of equality and the letters "Df" are to be regarded as one symbol, meaning jointly "is defined to mean." The sign of equality without the letters "Df" has a different meaning, to be defined shortly.

$$p \,.\, q \,.\!=.\, \sim(\sim p \vee \sim q) \quad \text{Df.}$$

This defines the logical product of two propositions p and q, *i. e.*, "p and q are both true." The above definition states that this is to mean: "It is false that either p is false or q is false." Here again, the definition does not give the only meaning which can be given to "p and q are both true," but gives the meaning which is most convenient for our purposes.

$$p \equiv q \,.\!=:\, p \supset q \,.\, q \supset p \quad \text{Df.}$$

That is, "$p \equiv q$," which is read "p is equivalent to q," means "p implies q and q implies p;" whence, of course, it follows that p and q are both true or both false.

$$(\exists x) \,.\, \phi x \,.\!=.\, \sim\{(x) \,.\, \sim \phi x\} \quad \text{Df.}$$

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* This sign, as well as the introduction of the idea which it expresses, are due to Frege. See his *Begriffsschrift* (Halle, 1879), p. 1, and *Grundgesetze der Arithmetik*, Vol. I (Jena, 1893), p. 9.

This defines "there is at least one value of x for which ϕx is true." We define it as meaning "it is false that ϕx is always false."

$$x = y \,.\, = : (\phi) : \phi ! x \,.\, \supset \,.\, \phi ! y \quad \text{Df.}$$

This is the definition of identity. It states that x and y are to be called identical when every predicative function satisfied by x is satisfied by y. It follows from the axiom of reducibility that if x satisfies ψx, where ψ is any function, predicative or non-predicative, then y satisfies ψy.

The following definitions are less important, and are introduced solely for the purpose of abbreviation.

$$(x, y) \,.\, \phi(x, y) \,.\, = : (x) : (y) \,.\, \phi(x, y) \quad \text{Df.}$$
$$(\exists x, y) \,.\, \phi(x, y) \,.\, = : (\exists x) : (\exists y) \,.\, \phi(x, y) \quad \text{Df.}$$
$$\phi x \,.\, \supset_x \,.\, \psi x : \; = : (x) : \phi x \supset \psi x \quad \text{Df.}$$
$$\phi x \,.\, \equiv_x \,.\, \psi x : \; = : (x) : \phi x \,.\, \equiv \,.\, \psi x \quad \text{Df.}$$
$$\phi(x, y) \,.\, \supset_{x, y} \,.\, \psi(x, y) : = : (x, y) : \phi(x, y) \,.\, \supset \,.\, \psi(x, y) \quad \text{Df.,}$$

and so on for any number of variables.

The primitive propositions required are as follows. (In 2, 3, 4, 5, 6, and 10, p, q, r stand for propositions.)

(1) A proposition implied by a true premise is true.

(2) $\vdash : p \vee p \,.\, \supset \,.\, p$.

(3) $\vdash : q \,.\, \supset \,.\, p \vee q$.

(4) $\vdash : p \vee q \,.\, \supset \,.\, q \vee p$.

(5) $\vdash : p \vee (q \vee r) \,.\, \supset \,.\, q \vee (p \vee r)$.

(6) $\vdash : . \, q \supset r \,.\, \supset : p \vee q \,.\, \supset \,.\, p \vee r$.

(7) $\vdash : (x) \,.\, \phi x \,.\, \supset \,.\, \phi y$;

i. e., "if all values of $\phi \hat{x}$ are true, then ϕy is true, where ϕy is any value."*

(8) If ϕy is true, where ϕy is any value of $\phi \hat{x}$, then $(x) \,.\, \phi x$ is true. This can not be expressed in our symbols; for if we write "$\phi y \,.\, \supset \,.\, (x) \,.\, \phi x$," that means "$\phi y$ implies that all values of $\phi \hat{x}$ are true, where y may have any value of the appropriate type," which is not in general the case. What we mean to assert is: "If, however y is chosen, ϕy is true, then $(x) \,.\, \phi x$ is true;" whereas what is expressed by "$\phi y \,.\, \supset \,.\, (x) \,.\, \phi x$" is: "However y is chosen, if ϕy is true, then $(x) \,.\, \phi x$ is true," which is quite a different statement, and in general a false one.

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* It is convenient to use the notation $\phi \hat{x}$ to denote the function itself, as opposed to this or that value of the function.

(9) $\vdash : (x) . \phi x . \supset . \phi a$, where a is any definite constant.

This principle is really as many different principles as there are possible values of a. *I. e.*, it states that, *e. g.*, whatever holds of all individuals holds of Socrates; also that it holds of Plato; and so on. It is the principle that a general rule may be applied to particular cases; but in order to give it scope, it is necessary to mention the particular cases, since otherwise we need the principle itself to assure us that the general rule that general rules may be applied to particular cases may be applied (say) to the particular case of Socrates. It is thus that this principle differs from (7); our present principle makes a statement about Socrates, or about Plato, or some other definite constant, whereas (7) made a statement about a variable.

The above principle is never used in symbolic logic or in pure mathematics, since all our propositions are general, and even when (as in "one is a number") we seem to have a strictly particular case, this turns out not to be so when closely examined. In fact, the use of the above principle is the distinguishing mark of *applied* mathematics. Thus, strictly speaking, we might have omitted it from our list.

(10) $\vdash :. (x) . p \vee \phi x . \supset : p . \vee . (x) . \phi x$;

i. e., "if 'p or ϕx' is always true, then either p is true, or ϕx is always true."

(11) When $f(\phi x)$ is true whatever argument x may be, and $F(\phi y)$ is true whatever possible argument y may be, then $\{f(\phi x) . F(\phi x)\}$ is true whatever possible argument x may be.

This is the axiom of the "identification of variables." It is needed when two separate propositional functions are each known to be always true, and we wish to infer that their logical product is always true. This inference is only legitimate if the two functions take arguments of the same type, for otherwise their logical product is meaningless. In the above axiom, x and y must be of the same type, because both occur as arguments to ϕ.

(12) If $\phi x . \phi x \supset \psi x$ is true for any possible x, then ψx is true for any possible x.

This axiom is required in order to assure us that the range of significance of ψx, in the case supposed, is the same as that of $\phi x . \phi x \supset \psi x . \supset . \psi x$; both are in fact the same as that of ϕx. We know, in the case supposed, that ψx is true whenever $\phi x . \phi x \supset \psi x$ and $\phi x . \phi x \supset \psi x . \supset . \psi x$ are both significant, but we do not know, without an axiom, that ψx is true whenever ψx is significant. Hence the need of the axiom.

Axioms (11) and (12) are required, *e. g.*, in proving

$$(x) . \phi x : (x) . \phi x \supset \psi x : \supset . (x) . \psi x.$$

By (7) and (11),

$$\vdash :. (x) . \phi x : (x) . \phi x \supset \psi x : \supset : \phi y . \phi y \supset \psi y$$

whence by (12),

$$\vdash :. (x) . \phi x : (x) . \phi x \supset \psi x : \supset : \psi y,$$

whence the result follows by (8) and (10).

(13) $\vdash :. (\exists f) :. (x) : \phi x . \equiv . f ! x.$

This is the axiom of reducibility. It states that, given any function $\phi\hat{x}$, there is a predicative function $f ! \hat{x}$ such that $f ! x$ is always equivalent to ϕx. Note that, since a proposition beginning with "$(\exists f)$" is, by definition, the negation of one beginning with "(f)," the above axiom involves the possibility of considering "all predicative functions of x." If ϕx is *any* function of x, we can not make propositions beginning with "(ϕ)" or "$(\exists \phi)$," since we can not consider "all functions," but only "*any* function" or "all *predicative* functions."

(14) $\vdash :. (\exists f) :. (x, y) : \phi(x, y) . \equiv . f ! (x, y).$

This is the axiom of reducibility for double functions.

In the above propositions, our x and y may be of any type whatever. The only way in which the theory of types is relevant is that (11) only allows us to identify real variables occurring in different contents when they are shown to be of the same type by both occurring as arguments to the same function, and that, in (7) and (9), y and a must respectively be of the appropriate type for arguments to $\phi\hat{z}$. Thus, for example, suppose we have a proposition of the form $(\phi) . f ! (\phi ! \hat{z}, x)$, which is a second-order function of x. Then by (7),

$$\vdash : (\phi) . f ! (\phi ! \hat{z}, x) . \supset . f ! (\psi ! \hat{z}, x),$$

where $\psi ! \hat{z}$ is any *first*-order function. But it will not do to treat $(\phi) . f ! (\phi ! \hat{z}, x)$ as if it were a first-order function of x, and take this function as a possible value of $\psi ! \hat{z}$ in the above. It is such confusions of types that give rise to the paradox of the *liar*.

Again, consider the classes which are not members of themselves. It is plain that, since we have identified classes with functions,* no class can be significantly said to be or not to be a member of itself; for the members of a class are arguments to it, and arguments to a function are always of lower type than the function. And if we ask: "But how about the class of all classes? Is not that a class, and so a member of itself?", the answer is twofold. First, if "the class of all classes" means "the class of all classes of whatever type," then there is no such notion. Secondly, if "the class of all classes" means "the class of all classes of type t," then this is a class of the next type above t, and is therefore again not a member of itself.

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* This identification is subject to a modification to be explained shortly.

Thus although the above primitive propositions apply equally to all types, they do not enable us to elicit contradictions. Hence in the course of any deduction it is never necessary to consider the absolute type of a variable; it is only necessary to see that the different variables occurring in one proposition are of the proper relative types. This excludes such functions as that from which our fourth contradiction was obtained, namely: "The relation R holds between R and S." For a relation between R and S is necessarily of higher type than either of them, so that the proposed function is meaningless.

VII.

Elementary Theory of Classes and Relations.

Propositions in which a function ϕ occurs may depend, for their truth-value, upon the particular function ϕ, or they may depend only upon the *extension* of ϕ, *i. e.*, upon the arguments which satisfy ϕ. A function of the latter sort we will call *extensional.* Thus, *e. g.*, "I believe that all men are mortal" may not be equivalent to "I believe that all featherless bipeds are mortal," even if men are coextensive with featherless bipeds; for I may not know that they are coextensive. But "all men are mortal" must be equivalent to "all featherless bipeds are mortal" if men are coextensive with featherless bipeds. Thus "all men are mortal" is an extensional function of the function "x is a man," while "I believe all men are mortal" is a function which is not extensional; we will call functions *intensional* when they are not extensional. The functions of functions with which mathematics is specially concerned are all extensional. The mark of an extensional function f of a function $\phi\,!\,\hat{z}$ is

$$\phi\,!\,x \,.\, \equiv_x \,.\, \psi\,!\,x : \supset_{\phi,\psi} : f(\phi\,!\,\hat{z}) \,.\, \equiv \,.\, f(\psi\,!\,\hat{z}).$$

From any function f of a function $\phi\,!\,\hat{z}$ we can derive an associated extensional function as follows. Put

$$f\{\hat{z}(\psi z)\} \,.\, = : (\exists\phi) : \phi\,!\,x \,.\, \equiv_x \,.\, \psi x : f\{\phi\,!\,\hat{z}\} \quad \text{Df.}$$

The function $f\{\hat{z}(\psi z)\}$ is in reality a function of $\psi\hat{z}$, though not the same function as $f(\psi\hat{z})$, supposing this latter to be significant. But it is convenient to treat $f\{\hat{z}(\psi z)\}$ technically as though it had an argument $\hat{z}(\psi z)$, which we call "the class defined by ψ." We have

$$\vdash :. \phi x . \equiv_x . \psi x : \supset : f\{\hat{z}(\phi z)\} . \equiv . f\{\hat{z}(\psi z)\},$$

whence, applying to the fictitious objects $\hat{z}(\phi z)$ and $\hat{z}(\psi z)$ the definition of identity given above, we find

$$\vdash :. \phi x . \equiv_x . \psi x : \supset . \hat{z}(\phi z) = \hat{z}(\psi z).$$

This, with its converse (which can also be proved), is the distinctive property of classes. Hence we are justified in treating $\hat{z}(\phi z)$ as the class defined by ϕ. In the same way we put

$$f\{\hat{x}\hat{y}\,\psi(x, y)\} . = : (\exists\phi) : \phi\,!(x, y) . \equiv_{x,y} . \psi(x, y) : f\{\phi\,!(\hat{x}, \hat{y})\} \quad \text{Df.}$$

A few words are necessary here as to the distinction between $\phi\,!(\hat{x}, \hat{y})$ and $\phi\,!(\hat{y}, \hat{x})$. We will adopt the following convention: When a function (as opposed to its values) is represented in a form involving $\hat{x}$ and $\hat{y}$, or any other two letters of the alphabet, the value of this function for the arguments a and b is to be found by substituting a for $\hat{x}$ and b for $\hat{y}$; *i. e.*, the argument mentioned first is to be substituted for the letter which comes earlier in the alphabet, and the argument mentioned second for the later letter. This sufficiently distinguishes between $\phi\,!(\hat{x}, \hat{y})$ and $\phi\,!(\hat{y}, \hat{x})$; *e. g.*:

The value of $\phi\,!(\hat{x}, \hat{y})$ for arguments a, b is $\phi\,!(a, b)$.
" " " " " b, a " $\phi\,!(b, a)$.
" " $\phi\,!(\hat{y}, \hat{x})$ " " a, b " $\phi\,!(b, a)$.
" " " " " b, a " $\phi\,!(a, b)$.

We put

$$x \epsilon \phi\,!\hat{z} . = . \phi\,!x \quad \text{Df.},$$

whence

$$\vdash :. x \epsilon \hat{z}(\psi z) . = : (\exists\phi) : \phi\,!y . \equiv_y . \psi y : \phi\,!x.$$

Also by the reducibility-axiom we have

$$(\exists\phi) : \phi\,!y . \equiv_y . \psi y,$$

whence

$$\vdash : x \epsilon \hat{z}(\psi z) . \equiv . \psi x.$$

This holds whatever x may be. Suppose now we want to consider $\hat{z}(\psi z)\,\varepsilon\hat{\phi}f\{\hat{z}(\phi\,!\,z)\}$. We have, by the above,

$$\vdash :.\, \hat{z}(\psi z)\,\varepsilon\hat{\phi}f\{\hat{z}(\phi\,!\,z)\} \,.\equiv:\, f\{\hat{z}(\psi z)\} :\equiv: (\exists\phi) : \phi\,!\,y \,.\equiv_y.\, \psi y : f\{\phi\,!\,z\},$$

whence

$$\vdash :.\, \hat{z}(\psi z) = \hat{z}(\chi z) \,.\supset:\, \hat{z}(\psi z)\varepsilon\varkappa \,.\equiv_{\varkappa}.\, \hat{z}(\chi z)\varepsilon\varkappa,$$

where $\varkappa$ is written for any expression of the form $\hat{\phi}f\{\hat{z}(\phi\,!\,z)\}$.

We put

$$cls = \hat{\alpha}\{(\exists\phi) \,.\, \alpha = \hat{z}(\phi\,!\,z)\} \quad \text{Df.}$$

Here *cls* has a meaning which depends upon the type of the apparent variable ϕ. Thus, *e. g.*, the proposition "*cls* ε *cls*," which is a consequence of the above definition, requires that "*cls*" should have a different meaning in the two places where it occurs. The symbol "*cls*" can only be used where it is unnecessary to know the type; it has an ambiguity which adjusts itself to circumstances. If we introduce as an indefinable the function "Indiv$\,!\,x$," meaning "x is an individual," we may put

$$Kl = \hat{\alpha}\{(\exists\phi) \,.\, \alpha = \hat{z}\,(\phi\,!\,z \,.\, \mathrm{Indiv}\,!\,z)\} \quad \text{Df.}$$

Then Kl is an unambiguous symbol meaning "classes of individuals."

We will use small Greek letters (other than ε, ϕ, ψ, χ, θ) to represent classes of whatever type; *i. e.*, to stand for symbols of the form $\hat{z}(\phi\,!\,z)$ or $\hat{z}(\phi z)$.

The theory of classes proceeds, from this point on, much as in Peano's system; $\hat{z}(\phi z)$ replaces $z\ni(\phi z)$. Also I put

$$\alpha \subset \beta \,.=:\, x\varepsilon\alpha \,.\supset_x.\, x\varepsilon\beta \quad \text{Df.}$$
$$\exists\,!\,\alpha \,.=.\, (\exists x) \,.\, x\varepsilon\alpha \quad \text{Df.}$$
$$V = \hat{x}(x = x) \quad \text{Df.}$$
$$\Lambda = x\{\sim(x = x)\} \quad \text{Df.}$$

where, Λ, as with Peano, is the null-class. The symbols $\exists$, Λ, V, like *cls* and ε, are ambiguous, and only acquire a definite meaning when the type concerned is otherwise indicated.

We treat relations in exactly the same way, putting

$$a\{\phi\,!\,(\hat{x}, \hat{y})\}b \,.=.\, \phi\,!\,(a, b) \quad \text{Df.}$$

(the order being determined by the alphabetical order of x and y and the typographical order of a and b); whence

$$\vdash :. a\{\hat{x}\hat{y}\psi(x, y)\}b . \equiv : (\exists\phi) : \psi(x, y) . \equiv_{x,y} . \phi\,!\,(x, y) : \phi\,!\,(a, b),$$

whence, by the reducibility-axiom,

$$\vdash : a\{\hat{x}\hat{y}\psi(x, y)\}b . \equiv . \psi(a, b).$$

We use Latin capital letters as abbreviations for such symbols as $\hat{x}\hat{y}\psi(x, y)$, and we find

$$\vdash :. R = S . \equiv : xRy . \equiv_{x,y} . xSy,$$

where

$$R = S . = : f\,!\,R . \supset_f . f\,!\,S \quad \text{Df.}$$

We put

$$\text{Rel} = \hat{R}\{(\exists\phi) . R = \hat{x}\hat{y}\phi\,!\,(x, y)\} \quad \text{Df.},$$

and we find that everything proved for classes has its analogue for dual relations. Following Peano, we put

$$\alpha \cap \beta = \hat{x}\,(x\varepsilon\alpha . x\varepsilon\beta) \quad \text{Df.},$$

defining the product, or common part, of two classes;

$$\alpha \cup \beta = \hat{x}\,(x\varepsilon\alpha . \vee . x\varepsilon\beta) \quad \text{Df.},$$

defining the sum of two classes; and

$$-\alpha = \hat{x}\,\{\sim(x\varepsilon\alpha)\} \quad \text{Df.},$$

defining the negation of a class. Similarly for relations we put

$$R \,\dot{\cap}\, S = \hat{x}\hat{y}\,(xRy . xSy) \quad \text{Df.}$$
$$R \,\dot{\cup}\, S = \hat{x}\hat{y}\,(xRy . \vee . xSy) \quad \text{Df.}$$
$$\dot{-} R = \hat{x}\hat{y}\,\{\sim(xRy)\} \quad \text{Df.}$$

VIII.

Descriptive Functions.

The functions hitherto considered have been propositional functions, with the exception of a few particular functions such $R \,\dot{\cap}\, S$. But the ordinary functions of mathematics, such as x^2, $\sin x$, $\log x$, are not propositional. Functions of this kind always mean "the term having such-and-such a relation to x." For this reason they may be called *descriptive* functions, because they *describe* a certain term by means of its relation to their argument. Thus "$\sin \pi/2$"

describes the number **1**; yet propositions in which sin $\pi/2$ occurs are not the same as they would be if 1 were substituted. This appears, *e. g.*, from the proposition "sin $\pi/2 = 1$," which conveys valuable information, whereas "$1 = 1$" is trivial. Descriptive functions have no meaning by themselves, but only as constituents of propositions; and this applies generally to phrases of the form "the term having such-and-such a property." Hence in dealing with such phrases, we must define any proposition in which they occur, not the phrases themselves.* We are thus led to the following definition, in which "$(\iota x)\,(\phi x)$" is to be read "*the* term x which satisfies ϕx."

$$\psi\,\{(\iota x)\,(\phi x)\}\,.\,=\,:\,(\exists b)\,:\,\phi x\,.\equiv_x.\,x = b\,:\,\psi b \quad \text{Df.}$$

This definition states that "the term which satisfies ϕ satisfies ψ" is to mean: "There is a term b such that ϕx is true when and only when x is b, and ψb is true." Thus all propositions about "*the* so-and-so" will be false if there are no so-and-so's or several so-and-so's.

The general definition of a descriptive function is

$$R\text{‘}y = (\iota x)\,(xRy) \quad \text{Df.};$$

that is, "$R\text{‘}y$" is to mean "the term which has the relation R to y." If there are several terms or none having the relation R to y, all propositions about $R\text{‘}y$ will be false. We put

$$E\,!\,(\iota x)\,(\phi x)\,.\,=\,:\,(\exists b)\,:\,\phi x\,.\equiv_x.\,x = b \quad \text{Df.}$$

Here "$E\,!\,(\iota x)\,(\phi x)$" may be read "there is such a term as the x which satisfies ϕx," or "the x which satisfies ϕx exists." We have

$$\vdash :.\, E\,!\,R\text{‘}y\,.\equiv:\,(\exists b)\,:\,xRy\,.\equiv_x.\,x = b.$$

The inverted comma in $R\text{‘}y$ may be read *of*. Thus if R is the relation of father to son, "$R\text{‘}y$" is "the father of y." If R is the relation of son to father, all propositions about $R\text{‘}y$ will be false unless y has one son and no more.

From the above it appears that descriptive functions are obtained from relations. The relations now to be defined are chiefly important on account of the descriptive functions to which they give rise.

$$\text{Cnv} = \hat{Q}\,\hat{P}\,\{xQy\,.\equiv_{x,y}.\,yPx\} \quad \text{Df.}$$

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*See the above-mentioned article "On Denoting," where the reasons for this view are given at length.

Here *Cnv* is short for "converse." It is the relation of a relation to its converse; *e. g.*, of *greater* to *less*, of parentage to sonship, of preceding to following, etc. We have

$$\vdash . \text{Cnv}`P = (\iota Q)\{xQy . \equiv_{x,y} . yPx\}.$$

For a shorter notation, often more convenient, we put

$$\breve{P} = \text{Cnv}`P \quad \text{Df.}$$

We want next a notation for the class of terms which have the relation R to y. For this purpose, we put

$$\overrightarrow{R} = \hat{\alpha}\hat{y}\{\alpha = \hat{x}(xRy)\} \quad \text{Df.},$$

whence

$$\vdash . \overrightarrow{R}`y = \hat{x}(xRy).$$

Similarly we put

$$\overleftarrow{R} = \hat{\beta}\hat{x}\{\beta = \hat{y}(xRy)\} \quad \text{Df.},$$

whence

$$\vdash . \overleftarrow{R}`x = \hat{y}(xRy).$$

We want next the *domain* of R (*i. e.*, the class of terms which have the relation R to something), the *converse domain* of R (*i. e.*, the class of terms to which something has the relation R), and the *field* of R, which is the sum of the domain and the converse domain. For this purpose we define the relations of the domain, converse domain, and field, to R. The definitions are:

$$D = \hat{\alpha}\hat{R}\{\alpha = \hat{x}((\exists y) . xRy)\} \quad \text{Df.}$$
$$Ɑ = \hat{\beta}\hat{R}\{\beta = \hat{y}((\exists x) . xRy)\} \quad \text{Df.}$$
$$C = \hat{\gamma}\hat{R}\{\gamma = \hat{x}((\exists y) : xRy . \vee . yRx)\} \quad \text{Df.}$$

Note that the third of these definitions is only significant when R is what we may call a *homogeneous* relation; *i. e.*, one in which, if xRy holds, x and y are of the same type. For otherwise, however we may choose x and y, either xRy or yRx will be meaningless. This observation is important in connection with Burali-Forti's contradiction.

We have, in virtue of the above definitions,

$$\vdash . D`R = \hat{x}\{(\exists y) . xRy\},$$
$$\vdash . Ɑ`R = \hat{y}\{(\exists x) . xRy\},$$
$$\vdash . C`R = \hat{x}\{(\exists y) : xRy . \vee . yRx\},$$

the last of these being significant only when R is homogeneous. "$D\text{‘}R$" is read "the domain of R;" "$\reflectbox{$D$}\text{‘}R$" is read "the converse domain of R," and "$C\text{‘}R$" is read "the field of R." The letter C is chosen as the initial of the word "campus."

We want next a notation for the relation, to a class α contained in the domain of R, of the class of terms to which some member of α has the relation R, and also for the relation, to a class β contained in the converse domain of R, of the class of terms which have the relation R to some member of β. For the second of these we put

$$R_\epsilon = \hat{\alpha}\hat{\beta}\,\{\alpha = \hat{x}\,((\exists y) \,.\, y \varepsilon \beta \,.\, xRy)\} \quad \text{Df.}$$

So that

$$\vdash .\, R_\epsilon\text{‘}\beta = \hat{x}\,\{(\exists y) \,.\, y \varepsilon \beta \,.\, xRy\}.$$

Thus if R is the relation of father to son, and β is the class of Etonians, $R_\epsilon\text{‘}\beta$ will be the class "fathers of Etonians;" if R is the relation "less than," and β is the class of proper fractions of the form $1 - 2^{-n}$ for integral values of n, $R_\epsilon\text{‘}\beta$ will be the class of fractions less than some fraction of the form $1 - 2^{-n}$; *i. e.*, $R_\epsilon\text{‘}\beta$ will be the class of proper fractions. The other relation mentioned above is $(\breve{R})_\epsilon$.

We put, as an alternative notation often more convenient,

$$R\text{“}\beta = R_\epsilon\text{‘}\beta \quad \text{Df.}$$

The *relative product* of two relations R, S is the relation which holds between x and z whenever there is a term y such that xRy and yRz both hold. The relative product is denoted by $R\,|\,S$. Thus

$$R\,|\,S = \hat{x}\hat{z}\,\{(\exists y) \,.\, xRy \,.\, yRz\} \quad \text{Df.}$$

We put also

$$R^2 = R\,|\,R \quad \text{Df.}$$

The product and sum of a class of classes are often required. They are defined as follows:

$$s\text{‘}\kappa = \hat{x}\,\{(\exists \alpha) \,.\, \alpha \varepsilon \kappa \,.\, x \varepsilon \alpha\} \quad \text{Df.}$$
$$p\text{‘}\kappa = \hat{x}\,\{\alpha \varepsilon \kappa \,.\supset_\alpha .\, x \varepsilon \alpha\} \quad \text{Df.}$$

Similarly for relations we put

$$\dot{s}\text{‘}\lambda = \hat{x}\hat{y}\,\{(\exists R) \,.\, R \varepsilon \lambda \,.\, xRy\} \quad \text{Df.}$$
$$\dot{p}\text{‘}\lambda = \hat{x}\hat{y}\,\{R \varepsilon \lambda \,.\supset_R .\, xRy\} \quad \text{Df.}$$

We need a notation for the class whose only member is x. Peano uses ιx, hence we shall use $\iota\text{'}x$. Peano showed (what Frege also had emphasized) that this class can not be identified with x. With the usual view of classes, the need for such a distinction remains a mystery; but with the view set forth above, it becomes obvious.

We put

$$\iota = \hat{\alpha}\hat{x}\,\{\alpha = \hat{y}\,(y = x)\} \quad \text{Df.},$$

whence

$$\vdash . \iota\text{'}x = \hat{y}\,(y = x),$$

and

$$\vdash : E\,!\,\breve{\iota}\text{'}\alpha . \supset . \breve{\iota}\text{'}\alpha = (\text{℩}x)\,(x \,\varepsilon\, \alpha);$$

i. e., if α is a class which has only one member, then $\breve{\iota}\text{'}\alpha$ is that one member.*

For the class of classes contained in a given class, we put

$$\text{Cl}\text{'}\alpha = \hat{\beta}\,(\beta \subset \alpha) \quad \text{Df.}$$

We can now proceed to the consideration of cardinal and ordinal numbers, and of how they are affected by the doctrine of types.

IX.

Cardinal Numbers.

The cardinal number of a class α is defined as the class of all classes *similar* to α, two classes being similar when there is a one-one relation between them. The class of one-one relations is denoted by $1 \rightarrow 1$, and defined as follows:

$$1 \rightarrow 1 = \hat{R}\,\{xRy . x'Ry . xRy' . \supset_{x,\,y,\,x',\,y'} . x = x' . y = y'\} \quad \text{Df.}$$

Similarity is denoted by *Sim*; its definition is

$$\text{Sim} = \hat{\alpha}\hat{\beta}\,\{(\exists R) . R \,\varepsilon\, 1 \rightarrow 1 . D\text{'}R = \alpha . \text{ᗡ}\text{'}R = \beta\}. \quad \text{Df.}$$

Then $\overrightarrow{\text{Sim}}\,\text{'}\alpha$ is, by definition, the cardinal number of α; this we will denote by $Nc\text{'}\alpha$; hence we put

$$Nc = \overrightarrow{\text{Sim}} \quad \text{Df.},$$

whence

$$\vdash . Nc\text{'}\alpha = \overrightarrow{\text{Sim}}\,\text{'}\alpha.$$

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*Thus $\breve{\iota}\text{'}\alpha$ is what Peano calls $\text{℩}\alpha$.

The class of cardinals we will denote by NC; thus

$$NC = Nc``cls \quad \text{Df.}$$

0 is defined as the class whose only member is the null-class, Λ, so that

$$0 = \iota`\Lambda \quad \text{Df.}$$

The definition of 1 is

$$1 = \hat{\alpha}\,\{(\exists c): x\varepsilon\alpha \,.\equiv_x.\, x = c\} \quad \text{Df.}$$

It is easy to prove that 0 and 1 are cardinals according to the definition.

It is to be observed, however, that 0 and 1 and all the other cardinals, according to the above definitions, are ambiguous symbols, like *cls*, and have as many meanings as there are types. To begin with 0: the meaning of 0 depends upon that of Λ, and the meaning of Λ is different according to the type of which it is the null-class. Thus there are as many 0's as there are types; and the same applies to all the other cardinals. Nevertheless, if two classes α, β are of different types, we can speak of them as having the same cardinal, or of one as having a greater cardinal than the other, because a one-one relation may hold between the members of α and the members of β, even when α and β are of different types. For example, let β be $\iota``\alpha$; *i. e.*, the class whose members are the classes consisting of single members of $\alpha$. Then $\iota``\alpha$ is of higher type than α, but similar to α, being correlated with α by the one-one relation ι.

The hierarchy of types has important results in regard to addition. Suppose we have a class of α terms and a class of β terms, where α and β are cardinals; it may be quite impossible to add them together to get a class of α and β terms, since, if the classes are not of the same type, their logical sum is meaningless. Where only a finite number of classes are concerned, we can obviate the practical consequences of this, owing to the fact that we can always apply operations to a class which raise its type to any required extent without altering its cardinal number. For example, given any class α, the class $\iota``\alpha$ has the same cardinal number, but is of the next type above α. Hence, given any finite number of classes of different types, we can raise all of them to the type which is what we may call the lowest common multiple of all the types in question; and it can be shown that this can be done in such a way that the resulting classes shall have no common members. We may then form the logical sum of all the classes so obtained, and its cardinal number will be the arithmetical sum of the cardinal numbers of the original classes. But where we have an infinite series of classes of ascending types, this method can not be applied. For this reason, we can not now prove that there must be infinite classes. For suppose there were only n individuals altogether in the universe, where n is finite. There would then be 2^n classes of individuals, and 2^{2^n} classes of classes of individuals, and so on. Thus the cardinal number of terms in each type would be finite; and though these numbers would grow beyond any assigned finite number, there would be no way of adding them so as to get an infinite number. Hence we need an axiom, so it would seem, to the effect that no finite class of individuals contains all individuals; but if any one chooses to assume that the total number of individuals in the universe is (say) 10,367, there seems no à priori way of refuting his opinion.

From the above mode of reasoning, it is plain that the doctrine of types avoids all difficulties as to the greatest cardinal. There is a greatest cardinal in each type, namely the cardinal number of the whole of the type; but this is always surpassed by the cardinal number of the next type, since, if α is the cardinal number of one type, that of the next type is 2^α, which, as Cantor has shown, is always greater than α. Since there is no way of adding different types, we can not speak of "the cardinal number of all objects, of whatever type," and thus there is no absolutely greatest cardinal.

If it is admitted that no finite class of individuals contains all individuals, it follows that there are classes of individuals having any finite number. Hence all finite cardinals exist as individual-cardinals; *i.e.*, as the cardinal numbers of classes of individuals. It follows that there is a class of $\aleph_0$ cardinals, namely, the class of finite cardinals. Hence $\aleph_0$ exists as the cardinal of a class of classes of classes of individuals. By forming all classes of finite cardinals, we find that $2^{\aleph_0}$ exists as the cardinal of a class of classes of classes of classes of individuals; and so we can proceed indefinitely. The existence of $\aleph_n$ for every finite value of n can also be proved; but this requires the consideration of ordinals.

If, in addition to assuming that no finite class contains all individuals, we assume the multiplicative axiom (*i.e.*, the axiom that, given a set of mutually exclusive classes, none of which are null, there is at least one class consisting of one member from each class in the set), then we can prove that there is a class of individuals containing $\aleph_0$ members, so that $\aleph_0$ will exist as an individual-cardinal. This somewhat reduces the type to which we have to go in order to prove the existence-theorem for any given cardinal, but it does not give us any existence-theorem which can not be got otherwise sooner or later.

Many elementary theorems concerning cardinals require the multiplicative axiom.[*] It is to be observed that this axiom is equivalent to Zermelo's,[†] and therefore to the assumption that every class can be well-ordered.[‡] These equivalent assumptions are, apparently, all incapable of proof, though the multiplicative axiom, at least, appears highly self-evident. In the absence of proof, it seems best not to assume the multiplicative axiom, but to state it as a hypothesis on every occasion on which it is used.

X.

Ordinal Numbers.

An ordinal number is a class of ordinally similar well-ordered series, *i. e.*, of relations generating such series. Ordinal similarity or *likeness* is defined as follows:

$$\mathrm{Smor} = \hat{P}\hat{Q}\,\{(\exists S) \,.\, S\varepsilon 1 \rightarrow 1 \,.\, \text{Ɑ}{\text{‘}}S = C{\text{‘}}Q \,.\, P = S \mid Q \mid \breve{S}\} \quad \text{Df.},$$

where "Smor" is short for "similar ordinally."

The class of serial relations, which we will call "Ser," is defined as follows:

$$\mathrm{Ser} = \hat{P}\,\{xPy \,.\supset_{x,y} .\, \sim(x = y) : xPy \,.\, yPz \,.\supset_{x,y,z} .\, xPz :$$
$$x\varepsilon C{\text{‘}}P \,.\supset_x .\, \overrightarrow{P}{\text{‘}}x \cup \iota{\text{‘}}x \cup \overleftarrow{P}{\text{‘}}x = C{\text{‘}}P\} \quad \text{Df.}$$

That is, reading P as "precedes," a relation is serial if (1) no term precedes itself, (2) a predecessor of a predecessor is a predecessor, (3) if x is any term in the field of the relation, then the predecessors of x together with x together with the successors of x constitute the whole field of the relation.

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[*] *Cf.* Part III of a paper by the present author, "On some Difficulties in the Theory of Transfinite Numbers and Order Types," *Proc. London Math. Soc.* Ser. II, Vol. IV, Part I.

[†] *Cf. loc. cit.* for a statement of Zermelo's axiom, and for the proof that this axiom implies the multiplicative axiom. The converse implication results as follows: Putting Prod ‘k for the multiplicative class of k, consider

$$Z{\text{‘}}\beta = \hat{R}\,\{(\exists x) \,.\, x\varepsilon\beta \,.\, D{\text{‘}}R = \iota{\text{‘}}\beta \,.\, \text{Ɑ}{\text{‘}}R = \iota{\text{‘}}x\} \quad \text{Df.},$$

and assume

$$\gamma\varepsilon\, \mathrm{Prod}\,{\text{‘}}Z{\text{‘‘}}cl{\text{‘}}a \,.\, R = \hat{\xi}\hat{x}\{(S) \,.\, \exists S\varepsilon\gamma \,.\, \xi S x\}.$$

Then R is a Zermelo-correlation. Hence if Prod ‘Z‘‘cl‘a is not null, at least one Zermelo-correlation for a exists.

[‡] See Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann." *Math. Annalen*, Vol. LIX, pp. 514–516.

Well-ordered serial relations, which we will call Ω, are defined as follows:

$$\Omega = \hat{P}\,\{P\varepsilon\,\mathrm{Ser} : \alpha \subset C\text{‘}P \,.\, \exists\,!\,\alpha \,.\supset_\alpha .\, \exists\,!\,(\alpha - \breve{P}\text{“}\alpha)\} \quad \mathrm{Df.};$$

i. e., P generates a well-ordered series if P is serial, and any class α contained in the field of P and not null has a first term. (Note that $\breve{P}\text{“}\alpha$ are the terms coming after some term of α).

If we denote by $No\text{‘}P$ the ordinal number of a well-ordered relation P, and by NO the class of ordinal numbers, we shall have

$$No = \hat{\alpha}\hat{P}\,\{P\varepsilon\Omega \,.\, \alpha = \overrightarrow{\mathrm{Smor}}\text{‘}P\} \quad \mathrm{Df.}$$
$$NO = No\text{“}\Omega.$$

From the definition of No we have

$$\vdash : P\varepsilon\,\Omega \,.\supset .\, No\text{‘}P = \overrightarrow{\mathrm{Smor}}\text{‘}P$$
$$\vdash : \sim(P\varepsilon\,\Omega) \,.\supset .\, \sim E\,!\,No\text{‘}P.$$

If we now examine our definitions with a view to their connection with the theory of types, we see, to begin with, that the definitions of "Ser" and Ω involve the *fields* of serial relations. Now the field is only significant when the relation is homogeneous; hence relations which are not homogeneous do not generate series. For example, the relation ι might be thought to generate series of ordinal number ω, such as

$$x,\ \iota\text{‘}x,\ \iota\text{‘}\iota\text{‘}x, \ldots\ldots \iota^n\text{‘}x, \ldots\ldots,$$

and we might attempt to prove in this way the existence of ω and $\aleph_0$. But x and $\iota\text{‘}x$ are of different types, and therefore there is no such series according to the definition.

The ordinal number of a series of individuals is, by the above definition of No, a class of relations of individuals. It is therefore of a different type from any individual, and can not form part of any series in which individuals occur. Again, suppose all the finite ordinals exist as individual-ordinals; *i. e.*, as the ordinals of series of individuals. Then the finite ordinals themselves form a series whose ordinal number is ω; thus ω exists as an ordinal-ordinal, *i. e.*, as the ordinal of a series of ordinals. But the type of an ordinal-ordinal is that of classes of relations of classes of relations of individuals. Thus the existence of ω has been proved in a higher type than that of the finite ordinals. Again, the cardinal number of ordinal numbers of well-ordered series that can be made out of finite ordinals is $\aleph_1$; hence $\aleph_1$ exists in the type of classes of classes of classes of relations of classes of relations of individuals. Also the ordinal numbers of well-ordered series composed of finite ordinals can be arranged in order of magnitude, and the result is a well-ordered series whose ordinal number is ω_1. Hence ω_1 exists as an ordinal-ordinal-ordinal. This process can be repeated any finite number of times, and thus we can establish the existence, in appropriate types, of $\aleph_n$ and ω_n for any finite value of n.

But the above process of generation no longer leads to any totality of *all* ordinals, because, if we take all the ordinals of any given type, there are always greater ordinals in higher types; and we can not add together a set of ordinals of which the type rises above any finite limit. Thus all the ordinals in any type can be arranged by order of magnitude in a well-ordered series, which has an ordinal number of higher type than that of the ordinals composing the series. In the new type, this new ordinal is not the greatest. In fact, there is no greatest ordinal in any type, but in every type all ordinals are less than some ordinals of higher type. It is impossible to complete the series of ordinals, since it rises to types above every assignable finite limit; thus although every segment of the series of ordinals is well-ordered, we can not say that the whole series is well-ordered, because the "whole series" is a fiction. Hence Burali-Forti's contradiction disappears.

From the last two sections it appears that, if it is allowed that the number of individuals is not finite, the existence of all Cantor's cardinal and ordinal numbers can be proved, short of $\aleph_\omega$ and ω_ω. (It is quite possible that the existence of these may also be demonstrable.) The existence of all *finite* cardinals and ordinals can be proved without assuming the existence of anything. For if the cardinal number of terms in any type is n, that of terms in the next type is 2^n. Thus if there are no individuals, there will be one class (namely, the null-class), two classes of classes (namely, that containing no class and that containing the null-class), four classes of classes of classes, and generally 2^{n-1} classes of the nth order. But we can not add together terms of different types, and thus we can not in this way prove the existence of any infinite class.

We can now sum up our whole discussion. After stating some of the paradoxes of logic, we found that all of them arise from the fact that an expression referring to *all* of some collection may itself appear to denote one of the collection; as, for example, "all propositions are either true or false" appears to be itself a proposition. We decided that, where this appears to occur, we are dealing with a false totality, and that in fact nothing whatever can significantly be said about *all* of the supposed collection. In order to give effect to this decision, we explained a doctrine of *types* of variables, proceeding upon the principle that any expression which refers to *all* of some type must, if it denotes anything, denote something of a higher type than that to all of which it refers. Where *all* of some type is referred to, there is an *apparent variable* belonging to that type. Thus *any expression containing an apparent variable is of higher type than that variable.* This is the fundamental principle of the doctrine of types. A change in the manner in which the types are constructed, should it prove necessary, would leave the solution of contradictions untouched so long as this fundamental principle is observed. The method of constructing types explained above was shown to enable us to state all the fundamental definitions of mathematics, and at the same time to avoid all known contradictions. And it appeared that in practice the doctrine of types is never relevant except where existence-theorems are concerned, or where applications are to be made to some particular case.

The theory of types raises a number of difficult philosophical questions concerning its interpretation. Such questions are, however, essentially separable from the mathematical development of the theory, and, like all philosophical questions, introduce elements of uncertainty which do not belong to the theory itself. It seemed better, therefore, to state the theory without reference to philosophical questions, leaving these to be dealt with independently.

Invariantive Reduction of Quadratic Forms in the $GF[2^n]$.[*]

By Leonard Eugene Dickson.

1. In the American Journal of Mathematics, Vol. XXI (1899), I gave a complete set of non-equivalent canonical forms of m-ary quadratic forms in the Galois field of order p^n. The cases $p = 2$ and $p > 2$ are essentially different. In the opening pages of the present paper, I give a simpler treatment of the important case $p = 2$, a treatment bringing to the front some of the invariants of the form. In §§ 4, 5, I show that the rank r of the discriminantal determinant gives the minimum number of variables on which the form can be expressed. The definition of r in this modular theory differs from that in the algebraic theory in the employment of the halves of the minors of odd order. In particular, for m odd, the discriminant vanishes identically in the $GF[2^n]$, while the semi-discriminant S_m is an important invariant.

The larger part of the paper is devoted to the determination and application of a complete set of linearly independent invariants of the ternary[†] quadratic form $a_1 x_2 x_3 + \dots + \Sigma b_i x_i^2$ in the $GF[2^n]$ for $n \leqq 4$. All the invariants may be expressed in terms of three fundamental independent invariants:

$$S_3 = a_1 a_2 a_3 + \Sigma a_i^2 b_i, \quad A = \prod_{i=1,2,3} (a_i^{2^n-1} - 1), \quad F = f + f^2 + f^4 + \dots + f^{2^{n-1}},$$

where f is a function increasing rapidly in complexity as n increases.

2. We consider the general m-ary quadratic form in the $GF[2^n]$:

$$Q_m(x) \equiv \sum_{i<j} c_{ij} x_i x_j + \Sigma b_i x_i^2 \qquad (i, j = 1, \dots, m). \qquad (1)$$

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[*] Presented before the American Mathematical Society (Chicago), Dec. 30, 1906.

[†] For the invariants of binary quadratic forms in the $GF[p^n]$, for both $p > 2$ and $p = 2$, see *Transactions American Math. Soc.*, Vol. VIII (1907), pp. 205–232.

For the invariants of m-ary quadratic forms in the $GF[2]$, *i. e.*, with $n = 1$, see *Proceedings London Math. Soc.*, Ser. 2, Vol. V (1907), pp. 301–324.

If every $c_{ij}=0$, we obtain the canonical form x_1^2, since every mark is a square and $\Sigma\, b_i x_i^2 = [\Sigma\, b_i^{\frac{1}{2}} x_i]^2$. In the contrary case, we may set $c_{12} \neq 0$. Then for

$$x_1' = x_1 + \sum_{i=3}^{m} c_{2i}\, x_i\,, \quad x_2' = c_{12}^{-1} x_2 + \sum_{i=3}^{m} c_{1i}\, x_i\,, \quad x_j' = c_{12}\, x_j\,(j>2),$$

$Q_m(x')$ reduces* to

$$x_1 x_2 + c_{12} \sum_{i<j}^{3,\ldots,m} [12\,ij]\, x_i x_j + b_1 x_1^2 + c_{12}^{-2} b_2 x_2^2 + \sum_{i=3}^{m} \beta_i x_i^2, \tag{2}$$

where $[12\,ij]$ denotes the Pfaffian $c_{12}\, c_{ij} - c_{1i}\, c_{2j} + c_{1j}\, c_{2i}$, and

$$\beta_i = c_{12}\, c_{1i}\, c_{2i} + b_1 c_{2i}^2 + b_2 c_{1i}^2 + b_i c_{12}^2. \tag{3}$$

For $m=3$, the vanishing of β_3 is a sufficient condition that (2) shall reduce to a binary form. It is also a necessary condition since, as shown below, β_3 is an invariant of Q_3. Similarly, for $m=4$, the vanishing of the invariant $[1234]$ is the necessary and sufficient condition that (2), and hence Q_4, shall be reducible to a ternary form.

Let next $m \geqq 5$. If every $[12\,ij]=0$, (2) is reducible to a ternary form. In the contrary case, we may set $[1234] \neq 0$ and remove the terms $x_3 x_i$, $x_4 x_i$ $(i>4)$ by a transformation which adds to x_3 and x_4 suitable linear functions of $x_5, \ldots, x_m$. Proceeding similarly, we conclude that either Q_m is expressible on fewer than m variables or else is reducible to

$$x_1 x_2 + x_3 x_4 + \ldots + x_{m-2} x_{m-1} + x_m^2 \qquad (m \text{ odd}), \tag{4}$$

$$x_1 x_2 + x_3 x_4 + \ldots + x_{m-1} x_m + \sum_{i=1}^{m} \delta_i x_i^2 \qquad (m \text{ even}). \tag{5}$$

The simple problem of the ultimate canonical forms of (5) is treated in § 6.

3. Although we shall derive independently (§ 4) the condition that Q_m shall reduce to a form in fewer than m variables, it seems worth while, in view of the peculiar character of the condition for m odd, to apply the preceding elementary method in the further examples $m=5$ and $m=6$.

When $m=5$, (2) is the sum of a binary form in x_1, x_2, and a ternary form in x_3, x_4, x_5. For the latter the ternary invariant (3) is c_{12}^2 times

$$c_{12}\,[1234]\,[1235]\,[1245] + \beta_3\,[1245]^2 + \beta_4\,[1235]^2 + \beta_5\,[1234]^2.$$

On inserting the values (3) of the β_i, we find that the coefficients of b_1 and b_2 equal $c_{12}^2\,[2345]^2$ and $c_{12}^2\,[1345]^2$, respectively, in view of the algebraic identity

$$c_{23}\,[1245] - c_{24}\,[1235] + c_{25}\,[1234] \equiv c_{12}\,[2345].$$

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* It is simpler to verify that, under the inverse transformation, (2) becomes $Q(x')$.

Further, the part independent of the b's is seen to equal $c_{12}^2\psi$, where

$$\psi = \sum_{(12)} c_{12}c_{13}c_{24}c_{35}c_{45} - \sum_{(10)} c_{12}^2 c_{34}c_{35}c_{45}, \tag{6}$$

where the first sum extends over the 12 products in which each subscript occurs exactly twice. Dropping the factor c_{12}^2, we obtain the invariant

$$\phi = \psi + b_1[2345]^2 + b_2[1345]^2 + \dots + b_5[1234]^2, \tag{7}$$

whose vanishing is the condition that Q_5 be reducible to a quaternary form.

For $m = 6$, the quaternary invariant for the terms $x_3, \dots, x_6$ of (2) is

$$[1234]\,[1256] - [1235]\,[1246] + [1236]\,[1245],$$

which is (algebraically) c_{12} times the Pfaffian $[123456]$.

4. The algebraic discriminant of the form (1) is

$$\Delta = \begin{vmatrix} 2b_1 & c_{12} & c_{13} & \dots & c_{1n} \\ c_{12} & 2b_2 & c_{23} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{1m} & c_{2m} & c_{3m} & \dots & 2b_m \end{vmatrix}.$$

In the $GF[2^n]$, this determinant is skew symmetric, and hence vanishes for m odd, while for m even it equals the square of the Pfaffian $[12\dots m]$.

For m odd, we define the *semi-discriminant* S_m of the form Q_m in the $GF[2^n]$ to be the expression derived *algebraically* by dividing by 2 each of the (even) coefficients in the expansion of Δ. Thus S_3 is β_3 and S_5 is ϕ, given by (3) and (7), respectively; indeed, Δ is congruent modulo 4 to $2\beta_3$ and 2ϕ, respectively.

Note that in Q_m any coefficient may be increased by a multiple of 2; but Δ is thereby increased by a multiple of 4, so that S_m is unaltered modulo 2.

All m-ary linear homogeneous transformations with coefficients in any given field F can be derived from generators of the two types:

$$x_1 = x_1' + t x_2', \quad x_i = x_i' \ (i > 1); \tag{8}$$

$$x_1 = \lambda x_1', \quad x_i = x_i' \ (i > 1). \tag{9}$$

Under these transformations Q becomes Q', with the (altered) coefficients:

$$b_2' = b_2 + tc_{12} + t^2 b_1, \quad c_{12}' = c_{12} + 2tb_1, \quad c_{2i}' = c_{2i} + tc_{1i} \quad (i = 3, \dots, m); \tag{8'}$$

$$b_1' = \lambda^2 b_1, \quad c_{1i}' = \lambda c_{1i} \quad (i = 2, \dots, m). \tag{9'}$$

For (9′), $\Delta' = \lambda^2\Delta$, since we may remove the factor λ from the first row and column. For (8′), Δ' becomes Δ if we subtract t times the elements of the first row from the second, and then subtract t times the elements of the first column from the second. From this formal algebraic result we conclude, in view of the remark in the preceding paragraph, that S_m is a relative invariant in the $GF[2^n]$. But $S_m \equiv 1$ for (4), $\Delta \equiv 1$ for (5), while $S_m \equiv 0$ and $\Delta \equiv 0$ for forms in fewer than m variables. Hence follows the

THEOREM: *According as* m *is even or odd, the vanishing of the (invariant) discriminant or semi-discriminant is the necessary and sufficient condition that an* m*-ary quadratic form in the* $GF[2^n]$ *shall be linearly transformable into a form of fewer than* m *variables.*

5. Suppose that, for m odd, S_m vanishes in the $GF[2^n]$, while not all the first minors M_{ij} of Δ vanish. Under a suitable linear transformation, Q_m becomes Q'_m, lacking the variable x_m. In the discriminant of Q'_m, the minor M'_{mm} alone does not vanish, since the M_{ij} are linear functions of it. Hence Q_m is expressible on $m-1$, but not on fewer, variables (§ 4).

Suppose that, for m even, the discriminant Δ vanishes in the $GF[2^n]$. Then all its first minors M_{ij} vanish. Indeed, $M_{ii} M_{jj} - M_{ij} M_{ji}$ is the product of Δ and a minor of degree $m-2$. But $M_{ii} \equiv M_{jj} \equiv 0 \pmod 2$, and $M_{ij} = M_{ji}$. Hence the M_{ij} may be assumed* to have the factor 2 algebraically, so that the *semi-minors* are unambiguously defined in the $GF[2^n]$. If the latter do not all vanish, Q_m is expressible on $m-1$, but not on fewer, variables (§ 4).

Combining our results, we obtain the

THEOREM: *In order that a quadratic form* Q_m *in the* $GF[2^n]$ *shall be reducible under linear transformation in the field to a quadratic form on* r *variables, but not reducible to one on less than* r *variables, it is necessary and sufficient that in the discriminantal determinant of* Q_m *every* $\mu^{(m)}, \ldots, \mu^{(r+1)}$ *shall vanish, but not every* $\mu^{(r)}$*, where* $\mu^{(s)}$ *ranges over the minors or semi-minors of order* s*, according as* s *is even or odd.*

6. It remains to complete the reduction of F_m, given by (5). We first reduce it to the form

$$x_1 x_2 + x_3 x_4 + \ldots + x_{m-1} x_m + x_1^2 + \delta x_2^2, \qquad \delta \equiv \delta_1 \delta_2 + \ldots + \delta_{m-1} \delta_m. \quad (5')$$

If every $\delta_i = 0$, no reduction is necessary. In the contrary case we may set $\delta_1 \neq 0$. Applying to $F_4(x''')$ in succession the three transformations:

$$x_1''' = \delta_1^{-\frac{1}{2}} x_1'', \; x_2''' = \delta_1^{\frac{1}{2}} x_2''; \quad x_1'' = x_1' + \delta_3^{\frac{1}{2}} x_3', \; x_4'' = x_4' + \delta_3^{\frac{1}{2}} x_2'; \quad x_3' = x_3 + \delta_4 x_4,$$

we obtain $x_1 x_2 + x_3 x_4 + x_1^2 + (\delta_1 \delta_2 + \delta_3 \delta_4) x_2^2$. Hence from F_m we reach (5').

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*In case $n > 1$, we first eliminate the n^{th} and higher powers of the root of the irreducible congruence (mod 2) defining the $GF[2^n]$.

Now* $x_1 x_2 + x_1^2 + \delta x_2^2$ is reducible in the $GF[2^n]$ if $\chi(\delta) = 0$, but is irreducible if $\chi(\delta) = 1$, where

$$\chi(\delta) \equiv \delta + \delta^2 + \delta^4 + \ldots + \delta^{2^{n-1}}. \tag{10}$$

The 2^{n-1} forms (5′) with $\chi(\delta) = 1$ are all equivalent,* but not reducible to one of the 2^{n-1} forms with $\chi(\delta) = 0$. The latter are evidently reducible to

$$x_1 x_2 + x_3 x_4 + \ldots + x_{m-1} x_m. \tag{11}$$

The forms (5) *constitute two non-equivalent classes characterized by the vanishing on non-vanishing of* $\chi(\delta)$, $\delta \equiv \delta_1 \delta_2 + \ldots + \delta_{m-1} \delta_m$.

It may be shown that $\chi(\delta)$ is an absolute invariant of the group of all linear transformations in the $GF[2^n]$ which preserve the *system* of forms (5).

7. We next seek a condition on the coefficients of the quadratic form Q_m (m even), of non-vanishing discriminant Δ in the $GF[2^n]$, which shall characterize à priori the class (§ 6) to which Q_m belongs. If $n = 1$, we have $\Delta = 1$ in the field. For any n, we shall assume, for the present, that $\Delta = 1$ (a slight normalization accomplished, for instance, by multiplying one of the variables by the mark $\Delta^{-\frac{1}{2}}$). In view of § 6, we may state our desiderata as follows: We seek a function ϕ of the coefficients of the form Q_m (m even) of discriminant unity, such that ϕ becomes $\chi(\delta)$ when Q_m specializes to (5), and such that ϕ is an absolute invariant of Q_m under the group of all m-ary linear homogeneous transformations of determinant unity in the $GF[2^n]$.

For $m = 2$, the problem is solved, since $\Delta = 1$ implies $c_{12} = 1$, whence

$$Q_2 = x_1 x_2 + b_1 x_1^2 + b_2 x_2^2, \quad \phi = \chi(b_1 b_2).$$

For $m = 4$, we apply to (5) the transformation (of determinant unity),

$$x_1 = \xi_1 + c_{23} \xi_3 + c_{24} \xi_4, \quad x_2 = \xi_2 + c_{13} \xi_3 + c_{14} \xi_4, \quad x_3 = \xi_3, \quad x_4 = \xi_4,$$

and obtain a form $Q_4(\xi)$ in which

$$c_{12} = 1, \quad c_{34} = 1 + c_{13} c_{24} + c_{14} c_{23}, \quad b_1 = \delta_1, \quad b_2 = \delta_2,$$
$$b_3 = \delta_3 + c_{13} c_{23} + \delta_1 c_{23}^2 + \delta_2 c_{13}^2, \quad b_4 = \delta_4 + c_{14} c_{24} + \delta_1 c_{24}^2 + \delta_2 c_{14}^2.$$

Hence by choice of the δ's the resulting form may be made identical with any form Q_4 in which $c_{12} = 1$, $[1234] = 1$. The last condition is equivalent to our assumption $\Delta = 1$ on Q_4. The restriction, $c_{12} = 1$, on the generality of Q_4 will be overcome by symmetry, as demanded by the invariance of ϕ. Expressing

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* AMERICAN JOURNAL, *l. c.*, p. 224; *Linear Groups*, p. 199.

$\delta_1\delta_2+\delta_3\delta_4$ in terms of the c_{ij}, b_i, and applying $c_{12}=1$, $[1234]=1$, we get $\psi+\rho+\rho^2$, where $\rho=b_1c_{23}c_{24}+b_2c_{13}c_{14}+c_{12}c_{34}$, and

$$\psi=\sum_{(6)}b_1b_2c_{34}^2+\sum_{(4)}b_1c_{23}c_{24}c_{34}+\sum_{(3)}c_{13}c_{14}c_{23}c_{24}. \tag{12}$$

Then $\chi(\delta_1\delta_2+\delta_3\delta_4)$ becomes $\chi(\psi)$ since $\chi(\rho+\rho^2)=0$ in the field. Now $\phi\equiv\chi(\psi)$ has the required properties. It remains only to show that ϕ is an absolute invariant of Q_4 under the group of all quaternary linear transformations of determinant unity. In view of the symmetry of (12), we may restrict the proof to the generator (8). Here (8′) becomes

$$b_2'=b_2+tc_{12}+t^2b_1,\quad c_{23}'=c_{23}+tc_{13},\quad c_{24}'=c_{24}+tc_{14}.$$

Under this transformation, the increment to ψ is

$$tb_1c_{34}[1234]+t^2b_1^2c_{34}^2+tc_{13}c_{14}[1234]+t^2c_{13}^2c_{14}^2,$$

and hence is of the form $\sigma+\sigma^2$, since $[1234]=1$. Hence the increment to $\phi=\chi(\psi)$ is $\chi(\sigma+\sigma^2)=0$, so that ϕ is an absolute invariant.

8. We next consider the determination of functions of the coefficients of Q_m which are invariant under every m-ary linear homogeneous transformation in the $GF[2^n]$.

As the independent invariants of Q_2 we may take*

$$c_{12},\quad (c_{12}^{2^n-1}-1)(b_1^{2^n-1}-1)(b_2^{2^n-1}-1),\quad \chi(b_1b_2c_{12}^{2^n-3}),$$

where χ is defined by (10). For $n=1$, we take $\chi(b_1b_2c_{12})$.

In the remainder of this paper we shall discuss Q_3 for low values of n.

9. Consider the ternary quadratic form in the $GF[2^n]$,

$$a_1x_2x_3+a_2x_1x_3+a_3x_1x_2+\Sigma b_ix_i^2. \tag{13}$$

We tabulate, for reference, a set of generators of the ternary linear group, and give the (altered) coefficients of the transformed quadratic form:

$$x_1=x_1'+tx_2':\quad a_1'=a_1+ta_2,\quad b_2'=b_2+t^2b_1+ta_3; \tag{14}$$

$$x_1=\lambda x_1':\quad a_2'=\lambda a_2,\quad a_3'=\lambda a_3,\quad b_1'=\lambda^2b_1; \tag{15}$$

$$(x_ix_j):\quad (a_ia_j)\,(b_ib_j). \tag{16}$$

We readily verify* the absolute invariance of

$$A=\Pi(a_i^{2^n-1}-1),\quad I=A\Pi(b_i^{2^n-1}-1)\qquad (i=1,2,3). \tag{17}$$

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* *Transactions Amer. Math. Soc.*, Vol. VIII (1907), pp. 514–522.

10. Let first $n = 1$, so that we consider the invariants of the ternary cubic (13) modulo 2. Let ϕ be a polynomial in the a's and b's with exponents 0 or 1. We may set

$$\phi = p + qa_1 + jb_2 + ka_1 b_2 \qquad (p,\ q,\ j,\ k \text{ independent of } a_1,\ b_2).$$

Now ϕ is invariant under (14), with $t = 1$, if and only if

$$a_2 \frac{\partial\phi}{\partial a_1} + (b_1 + a_3 \frac{\partial\phi}{\partial b_2} + a_2 (b_1 + a_3) \frac{\partial^2\phi}{\partial a_1 . \partial b_2} \equiv 0 \pmod 2.$$

The conditions are:

$$a_2 k \equiv 0, \quad (b_1 + a_3) k \equiv 0, \quad a_2 q + (b_1 + a_3) j \equiv 0. \tag{18}$$

From the first two,

$$k = (a_2 + 1) \{l(1 + b_1 + a_3) + mb_1 a_3\},$$

where l and m are (linear) functions of b_3 only. By subtracting from ϕ a suitable multiple of the invariant I, we may assume that m is independent of b_3. Hence no term of ϕ has the factor $a_1 b_2 . b_1 a_3 . b_3$. Applying the permutation [23], we see that no term of ϕ has the factor $a_1 b_2 a_2 b_1 b_3$. Hence l is independent of b_3. Applying the permutation [13] we obtain the terms with the factor $a_3 b_2$:

$$a_3 b_2 (a_2 + 1) \{l(1 + b_3 + a_1) + mb_3 a_1\}.$$

Hence those multiplying $a_1 b_2 a_3 (a_2 + 1)$ are $l + mb_3$. In the initial form of ϕ, the corresponding terms were $l + mb_1$. Hence $m \equiv 0$ and

$$k = l(a_2 + 1)(1 + b_1 + a_3), \qquad l = 0 \text{ or } 1.$$

In view of the terms multiplying $a_3 b_2$, we have

$$j = la_3 (a_2 + 1)(1 + b_3) + \alpha + \beta a_2 + \gamma b_1 + \delta a_2 b_1,$$

where α, β, γ, δ are functions of b_3 only. By (18_3), $(b_1 + a_3) j$ must have the factor a_2. Hence

$$\alpha \equiv \gamma \equiv l(b_3 + 1).$$

The terms of ϕ multiplying b_2 are $j + ka_1$. Hence those multiplying $b_1 b_2$ are $\gamma + \delta a_2 + la_1 (a_2 + 1)$. Since these must be symmetrical in a_1, a_2, we have $\delta \equiv l$. Set $\beta = \beta_1 + \beta_2 b_3$. Then the terms multiplying $b_2 b_3$ are:

$$la_3 (a_2 + 1) + l + \beta_2 a_2 + lb_1.$$

These must be symmetrical in a_2, a_3. Hence $\beta_2 \equiv l$, and

$$j = la_3 (a_2 + 1)(b_3 + 1) + l(b_1 + 1)(b_3 + 1) + la_2 (b_1 + b_3) + \beta_1 a_2.$$

The terms $j + ka_1$, which multiply b_2, may now be written in the form

$$lb_1 b_3 + lb_1 (a_1 + 1)(a_2 + 1) + lb_3 (a_2 + 1)(a_3 + 1)$$
$$+ l(a_1 + 1)(a_2 + 1)(a_3 + 1) + (l + \beta_1) a_2.$$

Those multiplying b_1 or b_3 may be obtained by symmetry. Hence

$$\phi = lK + (l + \beta_1) \Sigma a_i b_i + \psi,$$

where ψ is a function of a_1, a_2, a_3 only, while

$$K = b_1 b_2 b_3 + \Sigma b_i b_j (a_i + 1)(a_j + 1) + (\Sigma b_i) A \qquad (i, j = 1, 2, 3;\ i \neq j), \qquad (19)$$

$A = \Pi (a_i + 1)$ being the invariant (17) for $n = 1$. We may set

$$\psi = \lambda a_1 a_2 a_3 + \mu \Sigma a_i a_j + \nu \Sigma a_j.$$

Then the terms multiplying a_1, but not b_2, are:

$$q = l \{ b_1 b_3 (a_3 + 1) + (b_1 + b_3)(a_2 + 1)(a_3 + 1) + b_1 \}$$
$$+ \beta_1 b_1 + \lambda a_2 a_3 + \mu (a_2 + a_3) + \nu.$$

Then (18_3) gives $l + \beta_1 + \lambda + \mu \equiv 0$, $\mu \equiv \nu$. The invariant ϕ thus involves three arbitrary parameters l, λ, μ. Giving in turn one of these the value 1 and the other two the value 0, we obtain the invariants K and

$$S_3 = a_1 a_2 a_3 + \Sigma a_i b_i, \quad I_2 = \Sigma a_i b_i + \Sigma a_i a_j + \Sigma a_i,$$

S_3 occurring in § 4. Now $S_3 + I_2 + 1 = A$, while $K + A$ equals

$$J = \{ b_1 + (a_2 + 1)(a_3 + 1) \} \{ b_2 + (a_1 + 1)(a_3 + 1) \} \{ b_3 + (a_1 + 1)(a_2 + 1) \}. \qquad (20)$$

In the $GF[2]$ *the four linearly independent invariants of the ternary quadratic form* (13) *may be taken to be* A, I, S_3, J.

11. Let next $n = 2$, so that we consider the invariants of the ternary cubic (13) in the $GF[2^2]$. Under transformation (14), let a polynomial ϕ, with exponents $\leqq 3$, become ϕ'. We employ the abbreviations:

$$(1^i) = \frac{1}{i!} \frac{\partial^i \phi}{\partial a_1^i}, \quad (2^i) = \frac{1}{i!} \frac{\partial^i \phi}{\partial b_2^i}, \quad (1^i\, 2^j) = \frac{1}{i!\,j!} \frac{\partial^{i+j} \phi}{\partial a_1^i\, \partial b_2^j},$$

in which the division of the algebraic derivatives by $i!$ and $j!$ is to be performed algebraically and the quotients alone interpreted in the $GF[2^2]$. Then

$$\phi' - \phi = \tau_1 t + \tau_2 t^2 + \tau_3 t^3,$$

$$\tau_1 = a_2 (1) + a_3 (2) + b_1^2 (2^2) + a_2^2 b_1 (1^2\, 2) + a_3^2 b_1 (2^3) + a_2^3 a_3 (1^3\, 2)$$
$$+ a_2^2 a_3^2 (1^2\, 2^2) + (a_2 b_1^3 + a_2 a_3^3)(1\, 2^3) + a_2^3 b_1^2 (1^3\, 2^2) + a_2^2 a_3 b_1^2 (1^2\, 2^3)$$
$$+ a_2^3 a_3^2 b_1 (1^3\, 2^3),$$

$$\begin{aligned}\tau_2 = b_1 (2) + a_2^2 (1^2) + a_2 a_3 (1\ 2) + a_3^2 (2^2) + a_2 b_1^2 (1\ 2^2) + a_3 b_1^2 (2^3) \\ + a_2^3 b_1 (1^3\ 2) + a_2 a_3^2 b_1 (1\ 2^3) + a_2^3 a_3^2 (1^3\ 2^2) + (a_2^2 b_1^3 + a_2^2 a_3^3) (1^2\ 2^3) \\ + a_2^3 a_3 b_1^2 (1^3\ 2^3),\end{aligned}$$

$$\begin{aligned}\tau_3 = a_2 b_1 (1\ 2) + a_2^3 (1^3) + a_2^2 a_3 (1^2\ 2) + a_2 a_3^2 (1\ 2^3) + (a_3^3 + b_1^3) (2^3) \\ + a_2^2 b_1^2 (1^2\ 2^2) + a_2 a_3 b_1^2 (1\ 2^3) + a_2^2 a_3^2 b_1 (1^2\ 2^3) + (a_2^3 b_1^3 + a_2^3 a_3^3) (1^3\ 2^3).\end{aligned}$$

We may set

$$\phi = \sum_{i,j}^{0,1,2,3} A_{ij}\, a_1^i\, b_2^j \qquad (A_{ij} \text{ independent of } a_1, b_2).$$

When this expression is inserted, τ_1, τ_2, τ_3 must vanish* identically in a_1, b_2. From the coefficients of $b_2^3 a_1^2$, $b_2^2 a_1^3$, $b_2 a_1^3$, b_2^3, $b_2^2 a_1$, $b_2 a_1$ in τ_1, we get:

$$A_{33} a_2 = A_{33} a_3 = A_{33} b_1 = 0, \quad A_{13} a_2 = A_{13} a_3 = A_{13} b_1 = 0. \tag{21}$$

Hence must $A_{33} = \alpha\pi$, $A_{13} = \beta\pi$, where

$$\pi = (a_2^3 - 1)(a_3^3 - 1)(b_1^3 - 1),$$

while α and β are functions of b_3 only. Hence the factor of $a_1^3 a_3^3 b_2^3$ in ϕ is

$$\alpha(a_2^3 - 1)(b_1^3 - 1).$$

This must be symmetrical in b_1 and b_3. Hence $\alpha = \alpha_0 (b_3^3 - 1)$, where α_0 is a constant mark. Thus ϕ has the term

$$\alpha_0\, a_1^3 a_2^3 a_3^3 b_1^3 b_2^3 b_3^3,$$

which is unaltered by (15). If ϕ is not an absolute invariant, $\alpha_0 = 0$. If ϕ is an absolute invariant, we replace ϕ by $\phi - \alpha_0 I$, where I is the absolute invariant $(17)_{n=2}$. In either case, it remains to consider an invariant ϕ having $\alpha_0 = 0$. Since $A_{33} = 0$, ϕ has no term with the factor $a_1^3 b_2^3$. Applying suitable permutations of the subscripts, we conclude that

$$A_{33} = A_{13} = 0; \ \textit{no term of } \phi \textit{ has a factor } a_i^3\, b_j^3 \textit{ or } a_i\, b_j^3 \qquad (i \neq j). \tag{22}$$

From the coefficients of b_2^3, $a_1^2 b_2$, $a_1^2 b_2^2$, $a_1 b_2$, $a_1 b_2^2$ in τ_2 with $A_{33} = A_{13} = 0$, we get:

$$A_{23} a_2 = A_{23} a_3 = A_{23} b_1 = 0, \quad A_{31} a_2 = A_{32} a_2 = 0. \tag{23}$$

By the first three and (22),

$$A_{23} = 0; \ \textit{no term of } \phi \textit{ has a factor } a_i^2\, b_j^3 \qquad (i \neq j). \tag{24}$$

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* Note that $\tau_1 = 0$ does not imply $\tau_2 = 0$ as in the algebraic theory. In fact, for

$$\phi = a_1^3 a_2^3 a_3^3 + a_1 b_1 a_2^2 a_3^2 + a_2 b_2 a_1^2 a_3^2 + a_3 b_3 a_1^2 a_2^2,$$

$\tau_1 = 0$, but $\tau_2 \neq 0$.

With the simplifications $A_{13}=A_{23}=A_{33}=0$, $A_{31}a_2=A_{32}a_2=0$, we find as the conditions for $\tau_1=0$; $\tau_2=0$ (identically in a_1, b_2):

$$A_{31}a_3=A_{32}b_1^2,\quad A_{12}a_2=A_{03}a_3,\quad A_{11}a_2=A_{03}b_1^2, \tag{25}$$

$$A_{30}a_2+A_{21}a_3+A_{22}b_1^2=0,\quad A_{11}a_3=A_{12}b_1^2, \tag{26}$$

$$A_{10}a_2+A_{01}a_3+A_{02}b_1^2+A_{21}a_2^2b_1+A_{03}a_3^2b_1+A_{22}a_2^2a_3^2=0; \tag{27}$$

$$A_{32}a_3^2=A_{31}b_1,\quad A_{22}a_3^2=A_{21}b_1,\quad A_{03}a_3^2=A_{21}a_2^2, \tag{28}$$

$$A_{22}a_2^2=A_{03}b_1,\quad A_{30}a_2^2+A_{12}a_3^2+A_{11}b_1=0, \tag{29}$$

$$A_{20}a_2^2+A_{02}a_3^2+A_{12}a_2b_1^2+A_{01}b_1+A_{11}a_2a_3+A_{03}a_3b_1^2=0. \tag{30}$$

Finally, τ_3 becomes

$$(A_{11}a_2+A_{03}b_1^2)b_1+(A_{12}a_2+A_{03}a_3)a_3^2+(A_{30}a_2+A_{21}a_3+A_{22}b_1^2)a_2^2,$$

and hence is zero by (22) and (26_1). We multiply (28_2) by a_2^2 and apply to (27); we multiply (26_2) by a_2 and apply to (30); there result:

$$A_{10}a_2+A_{01}a_3+A_{02}b_1^2+A_{03}a_3^2b_1=0,\quad A_{20}a_2^2+A_{02}a_3^2+A_{01}b_1+A_{03}a_3b_1^2=0. \tag{31}$$

A polynomial ϕ, *lacking the highest term of* I, *will be an invariant if and only if it be unaltered by the simple transformations* (15), (16), *and satisfy conditions* (22), (23_4), (23_5), (24), (25), (26), (28), (29) *and* (31).

Denote the general term of ϕ by

$$a_1^{e_1}\,a_2^{e_2}\,a_3^{e_3}\,b_1^{f_1}\,b_2^{f_2}\,b_3^{f_3}. \tag{32}$$

The conditions that ϕ shall be unaltered by the transformations of type (15) are:

$$e_2+e_3+2f_1\equiv e_1+e_3+2f_2\equiv e_1+e_2+2f_3\equiv d \pmod 3, \tag{33}$$

where d is a fixed integer such that $\phi'=D^d\phi$ for a transformation of determinant D. We treat in turn the cases $d=0$, $d=1$, $d=2$.

12. Let first ϕ be an absolute invariant, so that $d\equiv 0$, and

$$f_1\equiv e_2+e_3,\quad f_2\equiv e_1+e_3,\quad f_3\equiv e_1+e_2 \pmod 3. \tag{33'}$$

For the terms $A_{32}a_1^3b_2^2$, $e_1=3$, $f_2=2$, so that $e_3=2$. By (23_5), a_2 occurs in A_{32} only in the combination a_2^3-1. Hence $e_2\equiv 0 \pmod 3$, $f_1=2$, $f_3=0$ or 3. By (22), the factor $a_1^3b_3^3$ does not occur. Hence

$$A_{32}=r\,a_3^2b_1^2(a_2^3-1),\quad r=\text{constant}. \tag{34}$$

Proceeding similarly with A_{31}, and determining the constant by either (25_1) or (28_1), we get

$$A_{31}=r\,a_3b_1(a_2^3-1). \tag{35}$$

Listing the possible terms (32) of A_{03}, A_{12}, A_{11}, in view of (33′), (22), (31), and imposing conditions (25_2), (25_3), (26_2), we readily find that:

$$A_{03}=\lambda a_2^3+\mu a_2b_1b_3+\nu a_2^2b_1^2b_3^2, \tag{36}$$

$$A_{12}=\lambda a_2^2a_3+\mu a_3b_1b_3+\nu a_2a_3b_1^2b_3^2, \tag{37}$$

$$A_{11}=\lambda a_2^2b_1^2+\mu b_1^3b_3+\nu a_2b_1b_3^2+l\,b_3(a_2^3-1)(a_3^3-1). \tag{38}$$

Applying conditions (26_1), (28_2), (28_3), (29), and requiring that the factor $\Sigma A_{3i} b_2^i$ of a_1^3 in ϕ shall be unaltered by [23], we find that:

$$l = \mu = \nu = r, \tag{39}$$

$$A_{30} = r(a_2 b_1 b_3 + a_2^2 b_1^2 b_3^2)(a_3^3 - 1) + s(a_2^3 - 1)(a_3^3 - 1) + \lambda a_2^3 a_3^3 + \lambda b_1^3, \tag{40}$$

$$A_{21} = r a_3^2 b_1^2 b_3^2 + r a_2^2 a_3^2 b_1 b_3 + \lambda a_2 a_3^2, \tag{41}$$

$$A_{22} = r b_3^2 (a_2^3 - 1)(a_3^3 - 1) + r b_1^3 b_3^2 + r a_2^2 b_1^2 b_3 + \lambda a_2 b_1. \tag{42}$$

Since the factors $\Sigma A_{1i} b_2^i$ and $\Sigma A_{2i} b_2^i$ of a_1 and a_1^2 must be unaltered by [23], while the factors $\Sigma A_{i1} a_1^i$ and $\Sigma A_{i2} a_1^i$ of b_2 and b_2^2 must be unaltered by [13]:

$$A_{10} = \lambda a_3^2 b_1^2 b_3 + \lambda a_2 a_3^2 b_3^2 + q a_2^2 a_3^2 b_1, \tag{43}$$

$$A_{20} = \lambda a_3 b_1 b_3^2 + \lambda a_2^2 a_3 b_3 + p a_2 a_3 b_1^2, \tag{44}$$

$$A_{01} = r a_3 b_1 + r a_3 b_1 b_3^3 + r a_2 a_3 b_1^2 b_3 + r a_2^3 a_3 b_1 + \lambda a_2^2 a_3 b_3^2, \tag{45}$$

$$A_{02} = r a_3^2 b_1^2 + r a_3^2 b_1^2 b_3^3 + r a_2^2 a_3^2 b_1 b_3^2 + r a_2^3 a_3^2 b_1^2 + \lambda a_2 a_3^2 b_3. \tag{46}$$

Conditions (31) require merely that

$$q = p = \lambda. \tag{47}$$

Since the terms $\Sigma A_{0i} b_2^i$, independent of a_1, must be unaltered by [23]; and the terms $\Sigma A_{i0} a_1^i$, independent of b_2, must be unaltered by [13]:

$$A_{00} = r(a_2 b_1 b_3 + a_2^2 b_1^2 b_3^2)(a_3^3 - 1) + s(a_2^3 - 1)(a_3^3 - 1) + \lambda a_3^3 b_3^3, \tag{48}$$

in which the constant term of ϕ has been taken to be s.

The A_{ij} have been so determined that ϕ is unaltered by the generators (14)–(16) of the ternary linear group in the $GF[2^2]$. Hence the resulting function ϕ is an absolute invariant. Of the parameters occurring in the above expressions for the A_{ij}, r, s and λ may be given arbitrary values in the field; while the remaining parameters are then determined by (39) and (47).

For $s = 1$, $r = \lambda = 0$, ϕ is the invariant A in (17) for $n = 2$.

For $\lambda = 1$, $r = s = 0$, $\phi = S_3^3$, where (§ 4)

$$S_3 = a_1 a_2 a_3 + a_1^2 b_1 + a_2^2 b_2 + a_3^2 b_3. \tag{49}$$

Finally, for $r = 1$, $s = \lambda = 0$, ϕ is the absolute invariant

$$\left.\begin{aligned} F = f + f^2, \quad f \equiv {} & a_1 b_1^3 b_2 b_3 + a_2 b_2^3 b_1 b_3 + a_3 b_3^3 b_1 b_2 + a_1 a_2 a_3 b_1^2 b_2^2 b_3^2 \\ & + a_1 a_2 b_1 b_2 b_3^2 + a_1 b_2 b_3 (a_2^3 - 1)(a_3^3 - 1) \\ & + a_1 a_3 b_1 b_3 b_2^2 + a_2 b_1 b_3 (a_1^3 - 1)(a_3^3 - 1) \\ & + a_2 a_3 b_2 b_3 b_1^2 + a_3 b_1 b_2 (a_1^3 - 1)(a_2^3 - 1). \end{aligned}\right\} \tag{50}$$

The four linearly independent absolute invariants of the ternary quadratic form (13) *in the* $GF[2^2]$ *may be taken to be* A *and* I, *given by* (17), S_3^3 *and* F, *given by* (49) *and* (50).

We note the relation $S_3 F = 0$.

13. We next readily prove that *the only relative invariants of* (13) *are* S_3 *and* S_3^2. It suffices to consider the case in which $d \equiv 2$ in (33). For, if $d \equiv 1$ and $\phi_1' = D\phi_1$, then $\phi' = D^2\phi$, where $\phi = \phi_1^2$. Since we shall prove that $\phi = S_3$, it follows that $\phi_1 = (\phi_1^2)^2 = S_3^2$.

Let therefore $d \equiv 2 \pmod 3$. Then, by (33),

$$f_1 \equiv e_2 + e_3 + 1, \quad f_2 \equiv e_1 + e_3 + 1, \quad f_3 \equiv e_1 + e_2 + 1. \tag{33''}$$

For the terms (32) of $A_{03} b_2^3$, $e_1 = 0$, $f_2 = 3$, so that $e_3 = 2$. But by (31), the factor $a_3^2 b_2^3$ cannot occur. Hence $A_{03} \equiv 0$. Then by (25_2), (25_3), (28_3), (29_1),

$$A_{12} a_2 = A_{11} a_2 = A_{21} a_2 = A_{22} a_2 = 0,$$

so that, in these A_{ij}, a_2 occurs only in the combination $a_2^3 - 1$, whence $e_2 \equiv 0 \pmod 3$. Hence for A_{12}, $e_1 = 1$, $f_2 = 2$, $e_3 \equiv 0 \pmod 3$, $f_1 = 1$, $f_3 = 2$. Hence A_{12} has the factor b_1. For A_{11}, $e_3 = 2$, $f_3 = 2$, $f_1 = 0, 3$; but $a_3^2 b_1^3$ is not a factor. Hence b_1 occurs in no term of A_{11}. Hence, by (26_2), $A_{11} \equiv A_{12} \equiv 0$. Similarly, A_{21} has the factor b_1^2, while b_1 occurs in no term of A_{22}; whence, by (28_2), $A_{21} \equiv A_{22} \equiv 0$.

In $A_{32} a_1^3 b_2^2$, $e_1 = 3$, $f_2 = 2$, so that $e_3 = 1$. But $a_3 b_2^2$ can not be a factor since $A_{12} \equiv 0$. Hence $A_{32} \equiv 0$.

By (25_1), (23) and (28_1), $A_{31} a_3 = A_{31} a_2 = A_{31} b_1 = 0$. Hence A_{31} has the factor π (§ 11), contrary to (22). Hence $A_{31} \equiv 0$.

By (26_1), $A_{30} a_2 = 0$, so that $e_2 \equiv 0 \pmod 3$. Hence $f_3 = 1$. But the factor $a_1^3 b_3$ can not occur since $A_{31} \equiv 0$. Hence $A_{30} \equiv 0$.

For A_{02}, $e_3 = 1$, whereas $a_3 b_2^2$ is not a factor. Hence $A_{02} \equiv 0$.

Since every $A_{i3} = A_{3i} = 0$, a factor a^3 or b^3 can not occur. Likewise, no factor $a_i b_j$, $a_i b_j^2$, $a_i^2 b_j$, $a_i^2 b_j^2$ $(i \neq j)$ can occur. It thus follows readily from (33'') that

$$A_{10} = \alpha a_2 a_3, \quad A_{01} = \beta b_1 b_3 + \gamma a_2^2, \quad A_{20} = \delta b_1, \quad A_{00} = \varepsilon a_3^2 b_3,$$

the last following since $e_3 = 2$, so that b_1 can not occur.

From (31), $\beta = 0$, $\gamma = \alpha$, $\delta = \alpha$. Applying the permutation [23], we get $\varepsilon = \alpha$. Hence $\phi = S_3$.

14. On comparing the invariants of the ternary quadratic form (13) in the $GF[2^n]$ for the cases $n=1$ and $n=2$, we note uniformity in the invariants A, I, S_3. In fact, these are invariants for any n. Corresponding to F in (50), there should be for $n=1$ an invariant analogous to f itself (compare § 6). We find that, for $n=1$, $I+J$ is precisely of the form f with the exponents omitted.

For $n=3$ the corresponding invariant must be of the form $f+f^2+f^4$. It would be natural to conjecture that f would be of the form (50) with exponents 3 changed to 7, and each a_i^1 changed to a_i^5 (in view of the weights). While the resulting terms do form a part of f, there occur 26 additional terms [see (91)].

15. We therefore proceed to investigate the invariants of the ternary form (13) in the $GF[2^3]$. Under transformation (14), let ϕ, with exponents $\overline{\leqq}\, 7$, become ϕ'. Using the same abbreviations as in § 11, we find that in $\phi'-\phi$ the coefficients of t, t^2, t^4 are, respectively:

$$\left.\begin{aligned}
&a_2(1)+a_3(2)+b_1^4(2^4)+a_2^2b_1^3(1^2\,2^3)+a_2^4b_1^2(1^4\,2^2)+a_2^3a_3b_1^2(1^3\,2^3)\\
&+a_3^4b_1^2(2^6)+a_2^6b_1(1^6\,2)+a_2^4a_3^2b_1(1^4\,2^3)+a_2^2a_3^4b_1(1^2\,2^5)+a_3^6b_1(2^7)\\
&+a_2^7a_3(1^7\,2)+a_2^6a_3^2(1^6\,2^2)+a_2^5a_3^3(1^5\,2^3)+a_2^4a_3^4(1^4\,2^4)+a_2^3a_3^5(1^3\,2^5)\\
&+a_2^2a_3^6(1^2\,2^6)+(a_2a_3^7+a_2b_1^7)(1\,2^7)+a_2^3b_1^6(1^3\,2^6)+a_2^2a_3b_1^6(1^2\,2^7)\\
&+a_2^5b_1^5(1^5\,2^5)+a_2^3a_3^2b_1^5(1^3\,2^7)+a_2^7b_1^4(1^7\,2^4)+a_2^6a_3b_1^4(1^6\,2^5)\\
&+a_2^5a_3^2b_1^4(1^5\,2^6)+a_2^4a_3^3b_1^4(1^4\,2^7)+a_2^5a_3^4b_1^3(1^7\,2^7)+a_2^7a_3^4b_1^2(1^7\,2^6)\\
&+a_2^6a_3^5b_1^2(1^6\,2^7)+a_2^7a_3^6b_1(1^7\,2^7),
\end{aligned}\right\}\quad(51)$$

$$\left.\begin{aligned}
&b_1(2)+a_2^2(1^2)+a_2a_3(1\,2)+a_3^2(2^2)+a_2b_1^4(1\,2^4)+a_3b_1^4(2^5)+a_2^3b_1^3(1^3\,2^3)\\
&+a_2^5b_1^2(1^5\,2^2)+a_2^4a_3b_1^2(1^4\,2^3)+a_2a_3^4b_1^2(1\,2^6)+a_3^5b_1^2(2^7)+a_2^7b_1(1^7\,2)\\
&+a_2^5a_3^2b_1(1^5\,2^3)+a_2^3a_3^4b_1(1^3\,2^5)+a_2a_3^6b_1(1\,2^7)+a_2^7a_3^2(1^7\,2^2)\\
&+a_2^6a_3^3(1^6\,2^3)+a_2^5a_3^4(1^5\,2^4)+a_2^4a_3^5(1^4\,2^5)+a_2^3a_3^6(1^3\,2^3)+a_2^2(a_3^7+b_1^7)(1^2\,2^7)\\
&+a_2^4b_1^6(1^4\,2^6)+a_2^3a_3b_1^6(1^3\,2^7)+a_2^6b_1^5(1^6\,2^5)+a_2^4a_3^2b_1^5(1^4\,2^7)+a_2^7a_3b_1^4(1^7\,2^5)\\
&+a_2^6a_3^2b_1^4(1^6\,2^6)+a_2^5a_3^3b_1^4(1^5\,2^7)+a_2^6a_3^4b_1^3(1^6\,2^7)+a_2^7a_3^5b_1^2(1^7\,2^7),
\end{aligned}\right\}\quad(52)$$

$$\left.\begin{aligned}
&b_1^2(2^2)+a_2^2b_1(1^2\,2)+a_3^2b_1(2^3)+a_2^4(1^4)+a_2^3a_3(1^3\,2)+a_2^2a_3^2(1^2\,2^2)\\
&+a_2a_3^3(1\,2^3)+a_3^4(2^4)+a_2b_1^5(1\,2^5)+a_2^3b_1^4(1^3\,2^4)+a_2^2a_3b_1^4(1^2\,2^5)\\
&+a_2a_3^2b_1^4(1\,2^6)+a_3^3b_1^4(2^7)+a_2^5b_1^3(1^5\,2^3)+a_2a_3^4b_1^3(1\,2^7)+a_2^7b_1^2(1^7\,2^2)\\
&+a_2^6a_3b_1^2(1^6\,2^3)+a_2^3a_3^4b_1^2(1^3\,2^6)+a_2^2a_3^5b_1^2(1^2\,2^7)+a_2^7a_3^2b_1(1^7\,2^3)\\
&+a_2^5a_3^4b_1(1^5\,2^5)+a_2^3a_3^6b_1(1^3\,2^7)+a_2^7a_3^4(1^7\,2^4)+a_2^6a_3^5(1^6\,2^5)+a_2^5a_3^6(1^5\,2^6)\\
&+a_2^4(a_3^7+b_1^7)(1^4\,2^7)+a_2^6b_1^6(1^6\,2^6)+a_2^5a_3b_1^6(1^5\,2^7)+a_2^6a_3^2b_1^5(1^6\,2^7)\\
&+a_2^7a_3^3b_1^4(1^7\,2^7).
\end{aligned}\right\}\quad(53)$$

We may set

$$\phi = \sum_{i,j}^{0,1,\ldots,7} A_{ij}\, a_1^i\, b_2^j \quad (A_{ij} \text{ independent of } a_1,\ b_2). \tag{54}$$

We require that (51) shall vanish identically in a_1, b_2 for this value of ϕ. The simplest of the resulting conditions are:

$$A_{ij} a_2 = A_{ij}\, a_3 = A_{ij}\, b_1 = 0 \quad (i, j = 1,7;\ 3,7;\ 5,7;\ 7,7;\ 5,5;\ 7,5). \tag{55}$$

(The remaining conditions are considered later.) For these six A_{ij},

$$A_{ij} = \alpha_{ij}\, \pi, \quad \pi \equiv (a_2^7 - 1)(a_3^7 - 1)(b_1^7 - 1),$$

where α_{ij} is a function of b_3 only. Hence the factor of $a_1^7 b_2^7 a_3^7$ in ϕ is

$$\alpha_{77}\, (a_2^7 - 1)\, (b_1^7 - 1).$$

This must be symmetrical in b_1 and b_3. Hence

$$\alpha_{77} = c\, (b_3^7 - 1), \quad c = \text{constant}.$$

On replacing ϕ by $\phi - c\, I$, where I is the absolute invariant given by (17) for $n = 3$, we may set $A_{77} \equiv 0$. Hence no term of ϕ can have a factor $a_i^7\, b_j^7\, (i \neq j)$. Thus

$$A_{17} = A_{37} = A_{57} = A_{77} = A_{55} = A_{75} = 0. \tag{56}$$

Among the conditions that (52) shall vanish, when (56) holds, are (55) for $i, j = 2,3;\ 2,7;\ 3,3;\ 6,3;\ 6,7;\ 7,3$, and

$$A_{76}\, a_2 = A_{76}\, a_3 = A_{36}\, a_2 = A_{35}\, a_2 = 0.$$

Two of the conditions from (51) now reduce to $A_{76}\, b_1 = 0$, $A_{35}\, a_3 = 0$. Hence in A_{35}, there would be the factor $(a_3^7 - 1)\, b_2^5$, whereas $A_{75} \equiv 0$. Hence

$$A_{23} = A_{27} = A_{33} = A_{63} = A_{67} = A_{73} = A_{76} = A_{35} = 0,\ A_{36}\, a_2 = 0. \tag{57}$$

When we apply (56) and (57) in computing (53), the conditions include:

$$A_{47}\, a_2 = A_{47}\, a_3 = A_{47}\, b_1 = 0, \quad A_{65}\, a_2 = A_{65}\, a_3 = 0, \quad A_{66}\, a_3 = A_{66}\, b_1 = 0,$$
$$A_{56}\, a_3 = A_{56}\, b_1 = 0, \quad A_{53}\, a_2 = A_{53}\, b_1 = 0, \quad A_{46}\, a_3 = A_{46}\, b_1 = 0.$$

Hence

$$A_{47} = A_{66} = A_{56} = A_{53} = A_{46} = A_{65} = 0. \tag{58}$$

In view of (57) and (58), certain of the conditions from (51) give

$$A_{36}\, b_1 = 0, \quad A_{15}\, a_2 = A_{15}\, a_3 = A_{15}\, b_1 = 0.$$

But $A_{36}\, a_2 = 0$ by (57). Hence

$$A_{36} = 0, \quad A_{15} = 0. \tag{59}$$

The A_{ij} in (56), (57), (58) and (59) are the only ones which vanish in every invariant distinct from I, as may be seen by examining S_3^7 and (91). We have therefore reached the limit to the simplification due to the vanishing A_{ij}. We next give the conditions on the non-vanishing A_{ij} which result from (51)–(53). In a few instances a multiple of the left member of one condition has been subtracted from that of a longer condition.

$$A_{71}a_2 = A_{72}a_2 = A_{74}a_2 = 0, \tag{60}$$

$$A_{71}a_3 + A_{74}b_1^4 = 0, \quad A_{71}b_1 + A_{72}a_3^2 = 0, \quad A_{72}b_1^2 + A_{74}a_3^4 = 0, \tag{61}$$

$$A_{16}a_2 + A_{07}a_3 = 0; \quad A_{13}a_2 + A_{07}b_1^4 = 0, \quad A_{13}a_3 + A_{16}b_1^4 = 0, \tag{62}$$

$$A_{51}a_3 + A_{54}b_1^4 = 0, \quad A_{51}a_2 + A_{45}b_1^4 = 0, \quad A_{54}a_2 + A_{45}a_3 = 0, \tag{63}$$

$$A_{34}a_2 + A_{25}a_3 = 0, \quad A_{31}a_2 + A_{25}b_1^4 = 0, \quad A_{31}a_3 + A_{34}b_1^4 = 0, \tag{64}$$

$$A_{43}a_3 + A_{52}a_2 = 0, \quad A_{14}a_2 + A_{05}a_3 = 0, \quad A_{32}a_2 + A_{26}b_1^4 = 0, \tag{65}$$

$$A_{70}a_2 + A_{61}a_3 + A_{64}b_1^4 = 0, \quad A_{50}a_2 + A_{41}a_3 + A_{44}b_1^4 = 0, \tag{66}$$

$$A_{12}a_2 + A_{03}a_3 + A_{06}b_1^4 = 0, \quad A_{11}a_2 + A_{05}b_1^4 + A_{42}a_2^4b_1^2 = 0, \tag{67}$$

$$A_{11}a_3 + A_{14}b_1^4 + A_{52}a_2^4b_1^2 = 0, \quad A_{30}a_2 + A_{24}b_1^4 + A_{21}a_3 + A_{62}a_2^4b_1^2 = 0, \tag{68}$$

$$A_{10}a_2 + A_{01}a_3 + A_{04}b_1^4 + A_{42}a_2^4b_1^2 + A_{43}a_2^4a_3^2b_1 = 0, \tag{69}$$

$$A_{43}b_1 + A_{62}a_2^2 = 0, \quad A_{43}a_3^2 + A_{61}a_2^2 = 0, \quad A_{61}b_1 + A_{62}a_3^2 = 0, \tag{70}$$

$$A_{13}b_1 + A_{32}a_2^2 = 0, \quad A_{31}a_2^2 + A_{13}a_3^2 = 0, \quad A_{31}b_1 + A_{32}a_3^2 = 0, \tag{71}$$

$$A_{07}b_1 + A_{26}a_2^2 = 0, \quad A_{07}a_3^2 + A_{25}a_2^2 = 0, \quad A_{45}b_1 + A_{64}a_2^2 = 0, \tag{72}$$

$$A_{03}b_1 + A_{22}a_2^2 + A_{07}a_3b_1^4 = 0, \quad A_{05}b_1 + A_{24}a_2^2 + A_{06}a_3^2 = 0, \tag{73}$$

$$A_{03}a_3^2 + A_{21}a_2^2 = 0, \quad A_{70}a_2^2 + A_{51}b_1 + A_{52}a_3^2 = 0, \tag{74}$$

$$A_{21}b_1 + A_{22}a_3^2 + A_{25}a_3b_1^4 = 0, \quad A_{11}b_1 + A_{12}a_3^2 + A_{30}a_2^2 = 0, \tag{75}$$

$$A_{34}a_2^2 + A_{16}a_3^2 = 0, \quad A_{41}b_1 + A_{42}a_3^2 + A_{60}a_2^2 + A_{51}a_2a_3 = 0, \tag{76}$$

$$A_{01}b_1 + A_{02}a_3^2 + A_{20}a_2^2 + A_{14}a_2b_1^4 + A_{52}a_2^5b_1^2 = 0, \tag{77}$$

$$A_{07}b_1^2 + A_{45}a_2^4 = 0, \quad A_{07}a_3^4 + A_{43}a_2^4 = 0, \quad A_{45}a_3^4 + A_{43}b_1^2 = 0, \tag{78}$$

$$A_{16}b_1^2 + A_{54}a_2^4 = 0, \quad A_{16}a_3^4 + A_{52}a_2^4 = 0, \quad A_{54}a_3^4 + A_{52}b_1^2 = 0, \tag{79}$$

$$A_{26}b_1^2 + A_{64}a_2^4 = 0, \quad A_{26}a_3^4 + A_{62}a_2^4 = 0, \quad A_{64}a_3^4 + A_{62}b_1^2 = 0, \tag{80}$$

$$A_{06}a_3^4 + A_{42}a_2^4 = 0, \quad A_{51}a_2^4 + A_{13}b_1^2 = 0, \quad A_{25}a_3^4 + A_{61}a_2^4 = 0, \tag{81}$$

$$A_{06}b_1^2 + A_{25}a_2^2b_1 + A_{44}a_2^4 = 0, \quad A_{03}b_1^2 + A_{05}a_3^4 + A_{41}a_2^4 = 0, \tag{82}$$

$$A_{60}a_2^4 + A_{22}b_1^2 + A_{24}a_3^4 = 0, \quad A_{70}a_2^4 + A_{32}b_1^2 + A_{34}a_3^4 = 0, \tag{83}$$

$$A_{42}b_1^2 + A_{43}a_3^2b_1 + A_{44}a_3^4 = 0, \quad A_{12}b_1^2 + A_{60}a_2^4 + A_{32}a_2^2a_3^2 + A_{14}a_3^4 = 0, \tag{84}$$

$$A_{02}b_1^2 + A_{21}a_2^2b_1 + A_{03}a_3^2b_1 + A_{40}a_2^4 + A_{04}a_3^4 + A_{22}a_2^2a_3^2 = 0. \tag{85}$$

16. Let ϕ be an absolute invariant whose general term has the notation (32), with exponents satisfying

$$e_2 + e_3 + 2f_1 \equiv e_1 + e_3 + 2f_2 \equiv e_1 + e_2 + 2f_3 \equiv 0 \pmod 7. \tag{86}$$

From (60) and (61) we get:

$$A_{71}=ra_3^5 b_1 (a_2^7-1), \quad A_{72}=ra_3^3 b_1^2 (a_2^7-1), \quad A_{74}=ra_3^6 b_1^4 (a_2^7-1).$$

Since the coefficient $\Sigma A_{7j} b_2^j$ of a_1^7 must be unaltered by [23],

$$A_{70}=r(a_2^3 b_1^2 b_3^2+a_2^5 b_1 b_3+a_2^6 b_1^4 b_3^4)(a_3^7-1)+ka_2^7 a_3^7+la_2^7+la_3^7+mb_1^7+c.$$

From (86), (62), (63), (78_1), (71), (64), (72), (80), (70_3), (79_2), (78_3):

$$\begin{aligned}
A_{07}&=\beta a_2^3 b_1^2 b_3^2+\gamma a_2^5 b_1 b_3+\delta a_2^6 b_1^4 b_3^4+\varepsilon a_2^7,\\
A_{16}&=\beta a_2^2 a_3 b_1^2 b_3^2+\gamma a_2^4 a_3 b_1 b_3+\delta a_2^5 a_3 b_1^4 b_3^4+\varepsilon a_2^6 a_3,\\
A_{13}&=\beta a_2^2 b_1^6 b_3^2+\gamma a_2^4 b_1^5 b_3+\delta a_2^5 b_1 b_3^4+\varepsilon a_2^6 b_1^4,\\
A_{45}&=\beta a_2^6 b_1^4 b_3^2+\gamma a_2 b_1^3 b_3+\delta a_2^2 b_1^5 b_3^4+\varepsilon a_2^3 b_1^2,\\
A_{54}&=\beta a_2^5 a_3 b_1^4 b_3^2+\gamma a_3 b_1^3 b_3+\delta a_2 a_3 b_1^6 b_3^4+\varepsilon a_2^2 a_3 b_1^2,\\
A_{51}&=\beta a_2^5 b_1 b_3^2+\gamma b_1^7 b_3+\delta a_2 b_1^3 b_3^4+\varepsilon a_2^2 b_1^6+sb_3(a_2^7-1)(a_3^7-1),\\
A_{31}&=\beta a_3^2 b_1^6 b_3^2+\gamma a_2^2 a_3^2 b_1^5 b_3+\delta a_2^3 a_3^2 b_1 b_3^4+\varepsilon a_2^4 a_3^2 b_1^4,\\
A_{34}&=\beta a_3^3 b_1^2 b_3^2+\gamma a_2^2 a_3^3 b_1 b_3+\delta a_2^3 a_3^3 b_1^4 b_3^4+\varepsilon a_2^4 a_3^3,\\
A_{25}&=\beta a_2 a_3^2 b_1^2 b_3^2+\gamma a_2^3 a_3^2 b_1 b_3+\delta a_2^4 a_3^2 b_1^4 b_3^4+\varepsilon a_2^5 a_3^2,\\
A_{32}&=\beta b_1^7 b_3^2+\gamma a_2^2 b_1^6 b_3+\delta a_2^3 b_1^2 b_3^4+\varepsilon a_2^4 b_1^5+pb_3^2(a_2^7-1)(a_3^7-1),\\
A_{26}&=\beta a_2 b_1^3 b_3^2+\gamma a_2^3 b_1^2 b_3+\delta a_2^4 b_1^5 b_3^4+\varepsilon a_2^5 b_1,\\
A_{62}&=\beta a_2^4 a_3^4 b_1^3 b_3^2+\gamma a_2^6 a_3^4 b_1^2 b_3+\delta a_3^4 b_1^5 b_3^4+\varepsilon a_2 a_3^4 b_1,\\
A_{64}&=\beta a_2^4 b_1^5 b_3^2+\gamma a_2^6 b_1^4 b_3+\delta b_1^7 b_3^4+\varepsilon a_2 b_1^3+qb_3^4(a_2^7-1)(a_3^7-1),\\
A_{61}&=\beta a_2^4 a_3^6 b_1^2 b_3^2+\gamma a_2^6 a_3^6 b_1 b_3+\delta a_3^6 b_1^4 b_3^4+\varepsilon a_2 a_3^6,\\
A_{52}&=\beta a_2^5 a_3^5 b_1^3 b_3^2+\gamma a_3^5 b_1 b_3+\delta a_2 a_3^5 b_1^4 b_3^4+\varepsilon a_2^2 a_3^5,\\
A_{43}&=\beta a_2^6 a_3^4 b_1^3 b_3^2+\gamma a_2 a_3^4 b_1 b_3+\delta a_2^2 a_3^4 b_1^4 b_3^4+\varepsilon a_2^3 a_3^4.
\end{aligned}$$

In view of (66_1), (74_2) and (83_2),

$$\beta=\gamma=\delta=q=s=p=r, \quad m=\varepsilon, \quad c=l, \quad k=\varepsilon+l. \tag{87}$$

Since the coefficient $A_{05}+A_{25}a_1^2+A_{45}a_1^4$ of b_2^5 in ϕ must be unaltered by the permutation [13], and likewise for the coefficient $A_{06}+A_{16}a_1+A_{26}a_1^2$ of b_2^6, and the coefficient $A_{03}+A_{13}a_1+A_{43}a_1^4$ of b_2^3, we get:

$$\begin{aligned}
A_{05}&=ra_2 a_3^4 b_1 b_3^3+ra_2^2 a_3^4 b_1^4 b_3^6+ra_2^3 a_3^4 b_1^2 b_3^4+\varepsilon a_2^3 a_3^4 b_3^2,\\
A_{06}&=ra_2^4 a_3^2 b_1^4 b_3^5+ra_2 a_3^2 b_1^2 b_3^3+ra_2^3 a_3^2 b_1 b_3^2+\varepsilon a_2^5 a_3^2 b_3,\\
A_{03}&=ra_2^2 a_3 b_1^2 b_3^6+ra_2^4 a_3 b_1 b_3^5+ra_2^5 a_3 b_1^4 b_3+\varepsilon a_2^6 a_3 b_3^4.
\end{aligned}$$

By (65_2), (67_2), (68_1):

$$\begin{aligned}
A_{14}&=ra_3^5 b_1 b_3^3+ra_2 a_3^5 b_1^4 b_3^6+ra_2^5 a_3^5 b_1^2 b_3^4+\varepsilon a_2^2 a_3^5 b_3^2,\\
A_{11}&=ra_3^4 b_1^5 b_3^3+ra_2 a_3^4 b_1 b_3^6+ra_2^4 a_3^4 b_1^3 b_3+(r+\varepsilon)a_2^2 a_3^4 b_1^4 b_3^2+\varepsilon a_2^6 a_3^4 b_1^2.
\end{aligned}$$

By (67_1), (73), (74_1), (81_1), (82):

$$\begin{aligned}
A_{12} &= ra_3^2 b_1^6 b_3^3 + ra_2 a_3^2 b_1^2 b_3^6 + ra_2^2 a_3^2 b_1^5 b_3^2 + (r+\varepsilon) a_2^4 a_3^2 b_1^4 b_3 + \varepsilon a_2^5 a_3^2 b_3^4,\\
A_{22} &= ra_3 b_1^3 b_3^6 + ra_2 a_3 b_1^6 b_3^2 + ra_2^2 a_3 b_1^2 b_3^5 + (r+\varepsilon) a_2^4 a_3 b_1 b_3^4 + \varepsilon a_2^5 a_3 b_1^4,\\
A_{24} &= ra_3^4 b_1^5 b_3^6 + ra_2^2 a_3^4 b_1^4 b_3^5 + ra_2^4 a_3^4 b_1^3 b_3^4 + (r+\varepsilon) a_2 a_3^4 b_1 b_3^2 + \varepsilon a_2^3 a_3^4 b_3,\\
A_{44} &= ra_3^2 b_1^6 b_3^5 + ra_2^2 a_3^2 b_1^5 b_3^4 + ra_2^4 a_3^2 b_1^4 b_3^3 + (r+\varepsilon) a_2 a_3^2 b_1^2 b_3 + \varepsilon a_2^3 a_3^2 b_1,\\
A_{41} &= ra_3 b_1^3 b_3^5 + ra_2 a_3 b_1^6 b_3 + ra_2^4 a_3 b_1 b_3^3 + (r+\varepsilon) a_2^2 a_3 b_1^2 b_3^4 + \varepsilon a_2^6 a_3 b_3^2,\\
A_{21} &= ra_3^3 b_1^2 b_3^6 + ra_2^2 a_3^3 b_1 b_3^5 + ra_2^3 a_3^3 b_1^4 b_3 + \varepsilon a_2^4 a_3^3 b_3^4,\\
A_{42} &= ra_3^6 b_1^4 b_3^5 + ra_2^4 a_3^6 b_1^2 b_3^3 + ra_2^6 a_3^6 b_1 b_3^2 + \varepsilon a_2 a_3^6 b_3.
\end{aligned}$$

By (66_2), (75_2), (83_1):

$$\begin{aligned}
A_{50} &= \varepsilon a_3^2 (b_1^6 b_3 + a_2 b_1^2 b_3^4 + a_2^2 b_1^5 + a_2^5 b_3^2),\\
A_{30} &= \varepsilon a_3^4 (b_1^5 b_3^2 + a_2^2 b_1^4 b_3 + a_2^4 b_1^3 + a_2^3 b_3^4),\\
A_{60} &= \varepsilon a_3 (b_1^3 b_3^4 + a_2^4 b_1 b_3^2 + a_2 b_1^6 + a_2^6 b_3).
\end{aligned}$$

It remains to determine A_{10}, A_{20}, A_{40}, A_{00}, A_{01}, A_{02}, A_{04}, which occur only in the three long conditions (69), (77) and (85). All the remaining conditions are seen to be now satisfied. Since the coefficient $\sum_{j=0}^{7} A_{ij} b_2^j$ of a_1^i in ϕ must be unaltered by [23], we get for $i = 1, 2, 4$:

$$\begin{aligned}
A_{10} &= \varepsilon a_3^3 (b_1^4 b_3^3 + a_2 b_3^6 + a_2^4 b_1^2 b_3) + \lambda a_2^6 a_3^6 b_1,\\
A_{20} &= \varepsilon a_3^5 (b_1 b_3^6 + a_2^2 b_3^5 + a_2 b_1^4 b_3^2) + \mu a_2^5 a_3^5 b_1^2,\\
A_{40} &= \varepsilon a_3^3 (b_1^2 b_3^5 + a_2^4 b_3^3 + a_2^2 b_1 b_3^4) + \nu a_2^3 a_3^3 b_1^4.
\end{aligned}$$

Since the terms $\Sigma A_{i0} a_1^i$ independent of b_2 are to be unaltered by [13],

$$\lambda = \mu = \nu = \varepsilon, \tag{88}$$

while the terms of A_{00} which involve a_3 are:

$$la_3^7 + la_2^7 a_3^7 + \varepsilon a_3^7 b_3^7 + ra_3^7 (a_2^5 b_1 b_3 + a_2^3 b_1^2 b_3^2 + a_2^6 b_1^4 b_3^4). \tag{89}$$

From the final conditions in (68), (77), (85), we now get:

$$\begin{aligned}
A_{01} &= \rho a_3^5 b_1 + \sigma a_3^5 b_1 b_3^7 + ra_2 a_3^5 b_1^4 b_3^3 + \varepsilon a_2^2 a_3^5 b_3^6 + ra_2^5 a_3^5 b_1^2 b_3 + \tau a_2^7 a_3^5 b_1,\\
A_{02} &= \rho a_3^3 b_1^2 + \sigma a_3^3 b_1^2 b_3^7 + ra_2^2 a_3^3 b_1 b_3^6 + \varepsilon a_2^4 a_3^3 b_3^5 + ra_2^3 a_3^3 b_1^4 b_3^2 + \tau a_2^7 a_3^3 b_1^2,\\
A_{04} &= \rho a_3^6 b_1^4 + \sigma a_3^6 b_1^4 b_3^7 + ra_2^4 a_3^6 b_1^2 b_3^5 + \varepsilon a_2 a_3^6 b_3^3 + ra_2^6 a_3^6 b_1 b_3^4 + \tau a_2^7 a_3^6 b_1^4.
\end{aligned}$$

Since the term $\rho a_3^5 b_1 b_2$ of $A_{01} b_2$ corresponds to $sa_1^5 b_2 b_3$ of $A_{51} a_1^5 b_2$, $\rho = s = r$. Similarly, we can determine σ, τ and the parameters of A_{00}. We can do this at one step by requiring that the terms $A_{0j} b_2^j$, independent of a_1, shall be unaltered by [23]. We find that

$$\rho = \sigma = \tau = r, \tag{90}$$

$$A_{00} = ra_2^3 b_1^2 b_3^2 + ra_2^5 b_1 b_3 + ra_2^6 b_1^4 b_3^4 + la_2^7 + l + \text{terms (89)},$$

where we have chosen the constant term of ϕ to be l.

In view of (87), (88) and (90), all the parameters occurring in the preceding expressions for the A_{ij} are expressed in terms of r, l, ε. That the remaining conditions (from the coefficients of a_1^6, a_1^5, a_1^3, b_2^4, b_2^2, b_2) for the invariance of ϕ under [13] and [23] are satisfied may be verified directly or from what follows. Hence ϕ is an absolute invariant with the three arbitrary parameters r, l, ε. Now l occurs only in A_{70} and A_{00}; thus for $l=1$, $r=\varepsilon=0$, ϕ is the invariant A given by (17). For $\varepsilon=1$, $r=l=0$, ϕ is seen to equal S_3^7, S_3 given by (49). Finally, for $r=1$, $l=\varepsilon=0$, ϕ becomes

$$\left.\begin{aligned}
F=f+f^2+f^4,\quad f\equiv&\sum_3 a_1^5 b_2 b_3 (a_2^7-1)(a_3^7-1)+\sum_3 a_1^5 b_1^7 b_2 b_3\\
&+\sum_3 a_1^5 a_2^5 b_1 b_2 b_3^2 \quad + a_1^5 a_2^5 a_3^5 b_1^2 b_2^2 b_3^2+\sum_3 a_1^5 a_2^5 a_3 b_1^4 b_2^4 b_3^2\\
&+\sum_6 a_1^5 a_2 b_1^3 b_2 b_3^4 \quad +\sum_3 a_1^5 a_2 a_3 b_1^6 b_2^4 b_3^4+\sum_\varepsilon a_1 a_2^2 b_1^6 b_2^3 b_3^2\\
&+\sum_3 a_1^2 a_2 a_3 b_1^6 b_2^2 b_3^2+\sum_3 a_1^4 a_2 a_3 b_1^6 b_2 b_3+a_1 a_2^2 a_3^4 b_1^4 b_2 b_3^2+a_1 a_3^2 a_2^4 b_1^4 b_3 b_2^2.
\end{aligned}\right\}\quad (91)$$

We note the relation $S_3 F=0$.

17. From the conditions in § 15, we readily determine the relative invariants. As in § 13, it suffices to treat the case $d\equiv 2$. We first prove that $A_{07}=0$, then that $A_{13}=A_{16}=0$, etc., proceeding as in § 13. The result found is that *the only relative invariants are the powers of* S_3.

18. The results for the cases $n=1, 2, 3$ differ only in the increasing complexity of the absolute invariant F. For $n=4$, the investigation was limited to the determination of this invariant. The parameters λ were restricted to the values 0, 1, so that $\lambda^2=\lambda$. Moreover, it was assumed that ϕ is identical with ϕ^2 in the $GF[2^4]$, so that the presence of any term implies the presence of its square, with the same coefficient (in view of $\lambda^2=\lambda$). Thus from A_{ij} we deduce $A_{2i,2j}$. The invariant thus determined is

$$F=f+f^2+f^4+f^8, \qquad (92)$$

where

$$\begin{aligned}
f=&\sum a_1^{13} b_2 b_3 (a_2^{15}-1)(a_3^{15}-1)+a_1^{13} a_2^{13} a_3^{13} b_1^2 b_2^2 b_3^2+\sum a_1^{13} b_1^{15} b_2 b_3+\sum a_1^{13} a_2^{13} b_1 b_2 b_3^2\\
&+\sum a_1^{13} a_2^{13} a_3 b_1^8 b_2^8 b_3^2+\sum a_1^{13} a_2 a_3 b_1^4 b_2^8 b_3^8+\sum a_1^{13} a_2 b_1^7 b_2 b_3^8+\sum a_1^{13} a_2^9 b_1^3 b_2 b_3^4\\
&+\sum a_1^{13} a_2^{13} a_3^9 b_1^4 b_2^4 b_3^2+\sum a_1^{13} a_2^9 a_3 b_1^{10} b_2^8 b_3^4+\sum a_1^{13} a_2^9 a_3^9 b_1^6 b_2^4 b_3^4+\sum a_1^{10} a_2 a_3 b_1^{14} b_2^2 b_3^2\\
&+\sum a_1^{10} a_2^{10} a_3 b_1^2 b_2^2 b_3^5+\sum a_1^{12} a_2 a_3 b_1^{14} b_2 b_3+\sum a_1^6 a_2 a_3 b_1^{14} b_2^4 b_3^4+\sum a_1^{12} a_2^{12} a_3 b_1 b_2 b_3^3\\
&+\sum a_1^6 a_2^6 a_3 b_1^4 b_2^4 b_3^9+\sum a_1^6 a_3 b_1^7 b_2^4 b_3^{12}+\sum a_1^{12} a_3 b_1^7 b_2 b_3^9+\sum a_1^6 a_2^5 a_3 b_1^{12} b_2^4 b_3^2\\
&+\sum a_1^6 a_2^4 a_3 b_1^5 b_2^4 b_3^{10}+\sum a_1^6 a_2^8 a_3 b_1^3 b_2^4 b_3^8+\sum a_1^{12} a_2^{10} a_3 b_1^2 b_2 b_3^4+\sum a_1^{12} a_2^8 a_3 b_1^3 b_2 b_3^5\\
&+\sum a_1^{10} a_2^8 a_3 b_1^3 b_2^2 b_3^6+\sum a_1^6 a_2^6 a_3^5 b_1^2 b_2^2 b_3^9+\sum a_1^6 a_2^5 a_3^5 b_1^{10} b_2^2 b_3^2+\sum a_1^6 a_2^5 b_1^5 b_2^{12} b_3^2\\
&+\sum a_1^8 a_2^5 a_3 b_1^{12} b_2^3 b_3+\sum a_1^8 a_2^4 a_3 b_1^5 b_2^3 b_3^9+\sum a_1^5 a_2^9 a_3 b_1^{10} b_2^{12} b_3^8+\sum a_1^{10} a_2 b_1^7 b_2^{10} b_3^2.
\end{aligned}$$

19. The invariants obtained for $n \leqq 4$ were expressed in terms of A, I, S_3 and F (F denoting $I+J$ when $n=1$). In each case we have noted the relation $S_3F=0$. Since every term of S_3 and F involves one or more a's, $AS_3=AF=0$. By inspection,

$$A^2=A,\ I^2=I,\ F^2=F,\ AI=I,\ IS_3=IF=0,\ S_3^{2^n}=S_3.$$

By means of these relations we may express the product of any two invariants as a linear function of A, I, F, S_3^i $(i=1, \ldots, 2^n-1)$. The latter may be taken as the units of a linear associative algebra with coördinates in the $GF[2^n]$.

20. *As a set of independent invariants we may take*

$$A,\ S_3,\ J\ (J=F+I).$$

Indeed, $AJ=I$. Next, to show that, for example, S_3 is independent of A and J, it suffices to exhibit two sets of values of the coefficients a_i, b_i, for which A has the same value, J the same value, but S_3 different values. Such a proof of the independence of A, S_3, J follows by inspection from the table below. From §§ 2, 6, we obtain a complete set of canonical forms of ternary quadratic forms in the $GF[2^n]$. In the second form, ρ is a particular solution of $\chi(\rho)=1$; for $n=1$ or 3, we may take $\rho=1$. We note that $F=\chi(f)$, where

$$f=\Sigma\, a_1^{\mu-2}\, b_2\, b_3\, (a_2^{\mu}-1)(a_3^{\mu}-1)+\text{terms with factor } b_1\, b_2\, b_3,$$

μ denoting 2^n-1. Thus $F=\chi(\rho^2)=1$ for the second form, $F=0$ for the others.

Canonical form	S_3	A	J
$x_1x_2+x_3^2$	1	0	0
$x_1x_2+\rho x_1^2+\rho x_2^2$	0	0	1
x_1x_2	0	0	0
x_1^2	0	1	0
Vanishing form	0	1	1

The five types are characterized by the respective sets of values:

$S_3=1$; $A=0, J=1$; $S_3=A=J=0$; $A=1, J=0$; $A=J=1$.

THE UNIVERSITY OF CHICAGO, *October*, 1906.

The Motion of a Particle Attracted Towards a Fixed Center by a Force Varying Inversely as the Fifth Power of the Distance.

By William Duncan MacMillan.

Introduction.

In his "*Traité des Fonctions Elliptiques*," Legendre discussed briefly a number of central forces, one of which was the very general one

$$f = \frac{A}{r^2} + \frac{B}{r^3} + \frac{C}{r^4} + \frac{D}{r^5}.$$

In an article, published in 1853, Stader* considered a number of others in rather more detail, among them the force

$$f = \frac{k^2}{r^5},$$

which has also been treated more recently by Miss Van Benschoten.† A short discussion of this law is given in many of the standard text books on Mechanics or Dynamics. The elliptic functions used in these discussions have been those of Jacobi and the treatment has been almost exclusively confined to orbits having apses.

In the present paper we shall make no restrictions upon the initial conditions, except that they shall be real. The different types of orbits are exhibited as a one-parameter set of curves. It is shown that for every orbit having a real apse there exists another orbit such that the radius vector of one is proportional to the reciprocal of the other, for all values of the true anomaly, and that on every orbit not having a real apse there exists a real point such that if the true anomaly be measured from this point the radius vector corresponding to a positive value of the anomaly is proportional to the reciprocal of the radius vector for the negative value of the same anomaly. There is developed also a relation between the times which is analogous to Kepler's Harmonic Law. These relations are established in a very direct and elegant manner by the use of the elliptic functions of Weierstrass, which we shall, consequently, adopt.

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* "De orbitis et motibus puncti cuiusdam corporei circa centrum attractionum aliis, quam Newtonia, attractionis legibus sollicitati," *Journal für Mathematik*, Vol. XLVI, p. 262.

† Master's Thesis in the Library of the University of Chicago.

The paper is divided into two parts: In part I the problem is studied from the standpoint of the theory of functions without regard to reality questions; in part II the cases of real orbits are discussed.

PART I.

§ 1. *Differential Equation of the Orbits.*

For the force, $f=\frac{k}{r^5}$, the differential equations in polar coordinates are

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2+\frac{k}{r^5}=0, \tag{1}$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0, \tag{2}$$

where k is a constant depending upon the units chosen and is positive or negative according as the force is attractive or repulsive. Equation (2) furnishes at once the integral of areas

$$r^2\frac{d\theta}{dt}=h. \tag{3}$$

We also find readily the *vis viva* integral

$$\left(\frac{dr}{dt}\right)^2=\tfrac{1}{2}k\frac{1}{r^4}-\frac{h^2}{r^2}+c_1. \tag{4}$$

From the relations

$$\frac{dr}{dt}=\frac{dr}{d\theta}\frac{d\theta}{dt}=\frac{h}{r^2}\frac{dr}{d\theta}$$

we obtain

$$\left(\frac{dr}{d\theta}\right)^2=\frac{k}{2h^2}-r^2+\frac{c_1}{h^2}r^4. \tag{5}$$

For arbitrary initial conditions

$$\theta = 0, \quad r = r_0, \quad \frac{d\frac{r}{r_0}}{d\theta} = \alpha, \tag{6}$$

this equation becomes

$$\left(\frac{d\frac{r}{r_0}}{d\theta}\right)^2 = \frac{k}{2h^2r_0^2} - \left(\frac{r}{r_0}\right)^2 + \left[1 + \alpha^2 - \frac{k}{2h^2r_0^2}\right]\left(\frac{r}{r_0}\right)^4;$$

or, putting

$$\begin{aligned} \frac{k}{2h^2r_0^2} &= \tfrac{1}{2}b, \\ 1 + \alpha^2 - \frac{k}{2h^2r_0^2} &= \tfrac{1}{2}\beta, \end{aligned} \tag{7}$$

we get the equation

$$\left(\frac{d\frac{r}{r_0}}{d\theta}\right)^2 = \tfrac{1}{2}b - \left(\frac{r}{r_0}\right)^2 + \tfrac{1}{2}\beta\left(\frac{r}{r_0}\right)^4 = R\left(\frac{r}{r_0}\right). \tag{8}$$

If now we make the transformation

$$r = \frac{1}{u}, \quad r_0 = \frac{1}{u_0},$$

we find

$$\left(\frac{d\frac{u}{u_0}}{d\theta}\right)^2 = \tfrac{1}{2}\beta - \left(\frac{u}{u_0}\right)^2 + \tfrac{1}{2}b\left(\frac{u}{u_0}\right)^4. \tag{9}$$

Equation (9) differs from equation (8) only in that b and β are interchanged. Consequently any solution for $\frac{r}{r_0}$ is also a solution for its reciprocal when b and β are interchanged.

§ 2. *General Solution of Differential Equation.*

Since only even powers of $\frac{r}{r_0}$ occur in equation (8), let us put

$$\left(\frac{r}{r_0}\right)^2 = z. \tag{10}$$

From this substitution there results

$$\left(\frac{dz}{d\theta}\right)^2 = 2\beta z\left[z - \frac{1}{\beta} - \frac{1}{\beta}\sqrt{1 - b\beta}\right]\left[z - \frac{1}{\beta} + \frac{1}{\beta}\sqrt{1 - b\beta}\right], \tag{11}$$

and this equation is reduced to the normal form of Weierstrass by putting

$$z = \frac{\frac{1}{2}b}{s + \frac{1}{3}};$$

whence

$$\left(\frac{ds}{d\theta}\right)^2 = 4\left[s + \tfrac{1}{3}\right]\left[s - \tfrac{1}{6} - \tfrac{1}{2}\sqrt{1 - b\beta}\right]\left[s - \tfrac{1}{6} + \tfrac{1}{2}\sqrt{1 - b\beta}\right] = R(s). \tag{12}$$

The roots of the equation $R(s) = 0$ are

$$\begin{aligned} e_\lambda &= \tfrac{1}{6} + \tfrac{1}{2}\sqrt{1 - b\beta}, \\ e_\mu &= -\tfrac{1}{3}, \\ e_\nu &= \tfrac{1}{6} - \tfrac{1}{2}\sqrt{1 - b\beta}; \end{aligned} \tag{13}$$

and

$$e_\lambda + e_\mu + e_\nu = 0.$$

The solution of equation (12) is

$$s = \wp(\theta + c'').$$

Consequently

$$z = \left(\frac{r}{r_0}\right)^2 = \frac{\frac{1}{2}b}{\wp(\theta + c'') - e_\mu}. \tag{14}$$

The reciprocal relation gives us also the solution

$$\left(\frac{u}{u_0}\right)^2 = \frac{\frac{1}{2}\beta}{\wp(\theta + c') - e_\mu}. \tag{15}$$

In comparing these two solutions it is to be observed that e_λ, e_μ and e_ν are unaltered by the interchange of b and β, so that the $\wp$ functions in the two cases have the same periods, these periods being functions of e_λ, e_μ and e_ν. It is more convenient to take the reciprocals of (14) and (15), which are

$$\begin{aligned} \left(\frac{r}{r_0}\right)^2 &= \frac{2}{\beta}\left[\wp(\theta + c') - e_\mu\right], \\ \left(\frac{u}{u_0}\right)^2 &= \frac{2}{b}\left[\wp(\theta + c'') - e_\mu\right]. \end{aligned} \tag{16}$$

Taking the product of these two expressions we find

$$\tfrac{1}{4}b\beta = \left[\wp(\theta + c'') - e_\mu\right]\left[\wp(\theta + c') - e_\mu\right],$$

and since this is true for all values of θ it defines the relation between c' and c''. If we take $\theta = -c''$ the first factor of the right member becomes infinite. Therefore the second factor must be zero, and consequently

$$c' - c'' = \omega_\mu,$$

where ω_μ is one of the half periods and $\wp(\omega_\mu) = e_\mu$.

If now we take

$$c' = c + \tfrac{1}{2}\,\omega_\mu, \qquad c'' = c - \tfrac{1}{2}\,\omega_\mu,$$

our solution becomes

$$\begin{aligned} \left(\frac{r}{r_0}\right)^2 &= \frac{2}{\beta}\,[\wp\,(\theta + c + \tfrac{1}{2}\,\omega_\mu) - e_\mu], \\ \left(\frac{u}{u_0}\right)^2 &= \frac{2}{b}\,[\wp\,(\theta + c - \tfrac{1}{2}\,\omega_\mu) - e_\mu]; \end{aligned} \tag{17}$$

and by virtue of the formula

$$8^{s}(\nu) - e_\mu = \left[\frac{\sigma_\mu}{\sigma}(\nu)\right]^2,$$

we have finally

$$\begin{aligned} \frac{r}{r_0} &= \sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}(\theta + c + \tfrac{1}{2}\,\omega_\mu), \\ \frac{u}{u_0} &= -\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}(\theta + c - \tfrac{1}{2}\,\omega_\mu). \end{aligned} \tag{18}$$

§ 3. *Apses, Zero-Points and Infinity Points.*

Let us put

$$\begin{aligned} \theta_\lambda &= -c - \tfrac{1}{2}\,\omega_\mu + \omega_\lambda, \\ \theta_\mu &= -c - \tfrac{1}{2}\,\omega_\mu + \omega_\mu, \\ \theta_\nu &= -c - \tfrac{1}{2}\,\omega_\mu + \omega_\nu, \end{aligned} \tag{19}$$

and denote the corresponding functional values by R_λ, R_μ, R_ν. We find then

$$\begin{aligned} \frac{R_\lambda}{r_0} &= \sqrt{\frac{2}{\beta}}\,\sqrt{e_\lambda - e_\mu} = \frac{\sqrt{\tfrac{1}{2}b}}{\sqrt{e_\nu - e_\mu}}, \\ \frac{R_\mu}{r_0} &= 0 \qquad\qquad\qquad = 0, \\ \frac{R_\nu}{r_0} &= \sqrt{\frac{2}{\beta}}\,\sqrt{e_\nu - e_\mu} = \frac{\sqrt{\tfrac{1}{2}b}}{\sqrt{e_\lambda - e_\mu}}. \end{aligned} \tag{20}$$

The derivative $\dfrac{dr}{d\theta}$ vanishes at the points $\theta = \theta_\lambda$ and $\theta = \theta_\nu$. These points are apses, while the point $\theta = \theta_\mu$ is a zero-point of the function. We may show this as follows: Put

$$\theta = \theta_\lambda + \tau.$$

Then (17) becomes

$$\left(\frac{r}{r_0}\right)^2 = \frac{2}{\beta}\,[\wp(\tau + \omega_\lambda) - e_\lambda], \tag{21}$$

and by means of the addition formula*

$$\wp(\tau + \omega_\lambda) - e_\lambda = \frac{(e_\mu - e_\lambda)(e_\nu - e_\lambda)}{\wp(\tau) - e_\lambda},$$

and the formula

$$\wp(\tau) - e_\lambda = \left[\frac{\sigma_\lambda}{\sigma}(\tau)\right]^2,$$

it is readily reduced to

$$\frac{r}{r_0} = \sqrt{\frac{2(e_\lambda - e_\mu)}{\beta}}\left[\frac{\wp(\tau) - e_\nu}{\wp(\tau) - e_\lambda}\right]^{\frac{1}{2}} = \sqrt{\frac{2(e_\lambda - e_\mu)}{\beta}}\,\frac{\sigma_\nu}{\sigma_\lambda}(\tau). \tag{22}$$

The σ quotient $\frac{\sigma_\nu}{\sigma_\lambda}(\tau)$ is an even function of τ; and therefore $r(\tau) = r(-\tau)$, and the point θ_λ is an apse. In a manner entirely similar it is shown that θ_ν is also an apse. For the point θ_μ, however, we find, by putting $\theta = \theta_\mu + \tau$,

$$\left(\frac{r}{r_0}\right)^2 = \frac{2}{\beta}\,[\wp(\tau + \omega_\mu) - \epsilon_\mu]\,; \tag{23}$$

and by the same formulæ as above this reduces to

$$\frac{r}{r_0} = \left[\frac{\frac{1}{2}b}{\wp(\tau) - e_\mu}\right]^{\frac{1}{2}} = \sqrt{\tfrac{1}{2}b}\,\frac{\sigma}{\sigma_\mu}(\tau), \tag{24}$$

which is an odd function of τ, vanishing for $\tau = 0$.

We have thus

$$\begin{aligned} \frac{r}{r_0}(\theta_\lambda + \tau) &= \frac{r}{r_0}(\theta_\lambda - \tau), \\ \frac{r}{r_0}(\theta_\mu + \tau) &= -\frac{r}{r_0}(\theta_\mu - \tau), \\ \frac{r}{r_0}(\theta_\nu + \tau) &= \frac{r}{r_0}(\theta_\nu - \tau). \end{aligned} \tag{25}$$

While the equations (17) have the periods $2\omega_\lambda$, $2\omega_\mu$, $2\omega_\nu$, equations (18) have the periods $4\omega_\lambda$, $4\omega_\mu$ and $4\omega_\nu$.†

From the formula

$$\frac{\sigma_\mu}{\sigma}(\theta \pm 2\omega_\lambda) = -\frac{\sigma_\mu}{\sigma}(\theta), \qquad \frac{\sigma_\mu}{\sigma}(\theta \pm 2\omega_\mu) = \frac{\sigma_\mu}{\sigma}(\theta),$$

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*See Schwarz, "*Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen,*" § 19, (5); also § 18, (2).

†Schwarz, § 23, (2).

it is readily verified that the points

$$\begin{array}{ll} \theta_\lambda + 2\omega_\lambda, & \theta_\nu + 2\omega_\lambda, \\ \theta_\lambda + 2\omega_\mu, & \theta_\nu + 2\omega_\mu, \\ \theta_\lambda + 2\omega_\nu, & \theta_\nu + 2\omega_\nu \end{array}$$

are all apses, while the points

$$\begin{array}{l} \theta_\mu + 2\omega_\lambda, \\ \theta_\mu + 2\omega_\mu, \\ \theta_\mu + 2\omega_\nu \end{array}$$

are all zero-points. In the complete period, therefore, there are 8 apses and 4 zero-points. The points for which θ equals

$$\begin{array}{l} -c - \tfrac{1}{2}\omega_\mu, \\ -c - \tfrac{1}{2}\omega_\mu + 2\omega_\lambda, \\ -c - \tfrac{1}{2}\omega_\mu + 2\omega_\mu, \\ -c - \tfrac{1}{2}\omega_\mu + 2\omega_\nu \end{array}$$

are four infinity points.

§ 4. *Middle Points.*

Returning to equations (18) let us take $\theta = -c$ and denote this value by ψ_0 and the corresponding value of r by R. If with these values we divide the first equation by the second, we find

$$\left(\frac{R}{r_0}\right)^2 = \sqrt{\frac{b}{\beta}}. \tag{26}$$

If we put $\theta = \psi_0 + \tau$ and consider negative values of τ in the second equation, they become

$$\begin{aligned} \frac{r}{r_0}(\psi_0 + \tau) &= \sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}(\tfrac{1}{2}\omega_\mu + \tau), \\ \frac{u}{u_0}(\psi_0 - \tau) &= \sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}(\tfrac{1}{2}\omega_\mu + \tau). \end{aligned} \tag{27}$$

Dividing the first of these equations by the second, we get

$$\left(\frac{r}{r_0}(\psi_0 + \tau)\right)\left(\frac{r}{r_0}(\psi_0 - \tau)\right) = \frac{R^2}{r_0^2},$$

or, simply,

$$r(\psi_0 + \tau) \,.\, r(\psi_0 - \tau) = R^2. \tag{28}$$

The point ψ_0 we have termed a middle point, and R is the middle distance.

Let us put

$$\psi_\lambda = -c + \omega_\lambda,$$
$$\psi_\mu = -c + \omega_\mu,$$
$$\psi_\nu = -c + \omega_\nu.$$

These points are also middle points. To show this let us put in (18) $\theta = \psi_\lambda + \tau$. We get

$$\begin{aligned} \frac{r}{r_0}(\psi_\lambda + \tau) &= \sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}(\quad \omega_\lambda + \tfrac{1}{2}\omega_\mu + \tau), \\ \frac{u}{u_0}(\psi_\lambda - \tau) &= -\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}(\quad \omega_\lambda - \tfrac{1}{2}\omega_\mu - \tau), \\ &= +\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}(-\omega_\lambda - \tfrac{1}{2}\omega_\mu - \tau), \\ &= -\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}(\quad \omega_\lambda + \tfrac{1}{2}\omega_\mu + \tau). \end{aligned} \tag{29}$$

Dividing, we find

$$\frac{r}{r_0}(\psi_\lambda + \tau)\,.\,\frac{r}{r_0}(\psi_\lambda - \tau) = -\frac{R^2}{r_0^2},$$

and similarly

$$\begin{aligned} \frac{r}{r_0}(\psi_\mu + \tau)\,.\,\frac{r}{r_0}(\psi_\mu - \tau) &= +\frac{R^2}{r_0^2}, \\ \frac{r}{r_0}(\psi_\nu + \tau)\,.\,\frac{r}{r_0}(\psi_\nu - \tau) &= -\frac{R^2}{r_0^2}. \end{aligned} \tag{30}$$

The addition of $2\omega_\lambda$, $2\omega_\mu$ or $2\omega_\nu$ to any of these points brings us to another middle point. In the complete parallelogram of periods there are thus 16 middle points.

Putting $\tau = \frac{1}{2}\omega_\lambda - \frac{1}{2}\omega_\nu$ in (28), we get

$$R_\lambda\, R_\nu = R^2, \tag{31}$$

which may be seen directly from equations (20). We find also, by putting $\gamma = b\beta$,

$$\begin{aligned} R_\lambda^2 &= \frac{1 + \sqrt{1-\gamma}}{\sqrt{\gamma}}\,R^2, \\ R_\nu^2 &= \frac{1 - \sqrt{1-\gamma}}{\sqrt{\gamma}}\,R^2. \end{aligned} \tag{32}$$

§ 5. *Reciprocal Orbits.*

If the initial conditions (equation (6)) be changed, that is, we consider another orbit which has the initial conditions

$$\bar{\theta} = 0, \qquad \bar{r} = \bar{r}_0, \qquad \frac{d\dfrac{\bar{r}}{\bar{r}_0}}{d\bar{\theta}} = -\alpha, \tag{33}$$

the constant of areas, h, being the same as before, and we determine $\bar{r}_0$ so that $\bar{r}_0 = \sqrt{\dfrac{b}{\beta}}\, r_0$, then

$$\frac{k}{2h^2\bar{r}_0^2} = \tfrac{1}{2}\beta, \qquad 1 + \alpha^2 - \frac{k}{2h^2\bar{r}_0^2} = \tfrac{1}{2}b \tag{34}$$

(b and β having the same quantitative values as before), and the differential equation of the orbit becomes

$$\left(\frac{d\dfrac{\bar{r}}{\bar{r}_0}}{d\bar{\theta}}\right)^2 = \tfrac{1}{2}\beta - \left(\frac{\bar{r}}{\bar{r}_0}\right) + \tfrac{1}{2}b\left(\frac{\bar{r}}{\bar{r}_0}\right)^4. \tag{35}$$

This equation is the same as (8), except that b and β are interchanged. It is indeed the same as equation (9). The solutions for $\dfrac{\bar{r}}{\bar{r}_0}$ and its reciprocal $\dfrac{\bar{u}}{\bar{u}_0}$ are obtained at once from (18) by interchanging b and β (e_λ, e_μ and e_ν remaining unchanged) and changing the signs of the solutions on account of the change in sign of the derivative. We have then

$$\begin{aligned} \frac{\bar{r}}{\bar{r}_0} &= -\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + \bar{c} + \tfrac{1}{2}\omega_\mu), \\ \frac{\bar{u}}{\bar{u}_0} &= +\sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + \bar{c} - \tfrac{1}{2}\omega_\mu). \end{aligned} \tag{36}$$

From the second of (36) and the first of (18) we have, by virtue of the initial conditions,

$$\sqrt{\frac{\beta}{2}} = \frac{\sigma_\mu}{\sigma}\,(\bar{c} - \tfrac{1}{2}\omega_\mu) = \frac{\sigma_\mu}{\sigma}\,(c + \tfrac{1}{2}\omega_\mu).$$

Therefore

$$\bar{c} = c + \omega_\mu.$$

Substituting this value in the first equation of (36) it becomes

$$\frac{\breve{r}}{\breve{r}_0} = -\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + c + \tfrac{3}{2}\,\omega_\mu), \tag{37}$$

or,

$$\frac{\breve{r}}{\breve{r}_0} = -\sqrt{\frac{2}{b}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + c - \tfrac{1}{2}\,\omega_\mu). \tag{38}$$

Comparing this with the second equation of (18), we have, when $\bar{\theta} = \theta$,

$$\frac{\breve{r}}{\breve{r}_0} = \frac{u}{u_0} = \frac{r_0}{r},$$

and consequently,

$$r\,\breve{r} = r_0\,\breve{r}_0 = R^2, \tag{39}$$

or,

$$\frac{r}{r_0} \cdot \frac{\breve{r}}{\breve{r}_0} = 1.$$

We will call this solution the reciprocal solution.

If in equation (37) we replace $\breve{r}_0$ by r_0, using the formula

$$\breve{r}_0 = \sqrt{\frac{b}{\beta}}\,r_0,$$

it becomes

$$\begin{aligned} \frac{\breve{r}}{r_0} &= -\sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + c + \tfrac{3}{2}\,\omega_\mu). \\ &= +\sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + c + \tfrac{3}{2}\,\omega_\mu + 2\,\omega_\lambda), \\ &= +\sqrt{\frac{2}{\beta}}\,\frac{\sigma_\mu}{\sigma}\,(\bar{\theta} + c + \tfrac{3}{2}\,\omega_\mu + 2\,\omega_\nu). \end{aligned} \tag{40}$$

Comparing this with the first equation of (18), we find

$$r\,(\bar{\theta} + \omega_\mu + 2\,\omega_\lambda) = r\,(\bar{\theta} + \omega_\mu + 2\,\omega_\nu) = \breve{r}\,(\bar{\theta}). \tag{41}$$

The reciprocal solution is thus included as a branch of the original solution.

§ 6. *Velocity in Orbit.*

The expression for the velocity in orbit is given by

$$v^2 = \tfrac{1}{2}\,\frac{k}{r^4} + c_1,$$

which is the same as equation (4). The value of c_1, as there determined by the initial conditions, is

$$c_1 = \tfrac{1}{2} \cdot \frac{\beta}{b\,r_0^4}\,k = \frac{k}{2\,R^4}.$$

Substituting this value of c_1, we have

$$v^2 = \tfrac{1}{2} k \left(\frac{1}{R^4} + \frac{1}{r^4}\right). \tag{42}$$

At the middle point ψ_0,

$$v^2_{\psi_0} = \frac{k}{R^4}.$$

§ 7. *Tangents and Asymptotes.*

Denoting by ϕ the angle between the radius vector and the tangent to the curve, we have from the calculus

$$\tan \phi = \frac{r}{\dfrac{dr}{d\theta}}.$$

Hence

$$\tan^2 \phi = \frac{\left(\dfrac{r}{r_0}\right)^2}{\tfrac{1}{2} b - \left(\dfrac{r}{r_0}\right)^2 + \tfrac{1}{2} \beta \left(\dfrac{r}{r_0}\right)^4}, \tag{43}$$

which reduces without difficulty to

$$\tfrac{1}{2} b \, r_0^2 r^2 \left[\frac{1}{R^4} + \frac{1}{r^4}\right] \sin^2 \phi = 1, \tag{44}$$

or, as may also be written,

$$r^2 v^2 \sin^2 \phi = h^2,$$

or,

$$\tfrac{1}{2} \sqrt{\gamma} \left[\frac{r^2}{R^2} + \frac{R^2}{r^2}\right] \sin^2 \phi = 1.$$

Since $r \sin \phi = p$ is the perpendicular from the origin to the tangent, we have

$$v = \frac{h}{p}.$$

The derivative of $\tan \phi$ with respect to r vanishes for $\left(\dfrac{r}{r_0}\right)^4 = \dfrac{b}{\beta}$; that is, at the middle points.

The expression for the polar subtangent,

$$P = \frac{r_0 \left(\dfrac{r}{r_0}\right)^2}{\sqrt{\tfrac{1}{2} b - \left(\dfrac{r}{r_0}\right)^2 + \tfrac{1}{2} \beta \left(\dfrac{r}{r_0}\right)^4}},$$

has, as $\frac{r}{r_0} \doteq \infty$, the finite limiting value $\frac{r_0}{\sqrt{\frac{1}{2}\beta}}$, or, which is the same thing, $\frac{\sqrt{2}\,R}{\sqrt[4]{\gamma}}$. This limiting value of the polar subtangent is the perpendicular distance from the origin to the asymptote. Consequently every infinite branch has an asymptote.

§ 8. *Determination of the Time.*

From the integral of areas,

$$r^2 \frac{d\theta}{dt} = h,$$

we have, by substituting the value of r^2 from (17),

$$\frac{2}{\beta}\left[\wp(\theta + c + \tfrac{1}{2}\omega_\mu) - e_\mu\right] d\theta = \frac{h}{r_0^2}\, dt, \tag{45}$$

which can be integrated directly, since the $\wp$ function is the negative derivative of the ζ function.

Integrating and determining the constant so that θ vanishes with t, we get

$$\zeta(c + \tfrac{1}{2}\omega_\mu) - \zeta(\theta + c + \tfrac{1}{2}\omega_\mu) - e_\mu\,\theta = \frac{\beta h}{2 r_0^2}\, t. \tag{46}$$

Consider now another orbit (the symbols for which we will denote by the subscripts 1) such that

$$\begin{aligned} \alpha &= \alpha_1, \\ h r_0 &= h_1 r_1. \end{aligned} \tag{47}$$

Then we will have also

$$\begin{aligned} b &= b_1, \\ \beta &= \beta_1, \\ \gamma &= \gamma_1, \\ \frac{R}{r_0} &= \frac{R_1}{r_1}, \\ c &= c_1. \end{aligned}$$

Under these conditions the orbits will be similar, but will differ in size. The expression for the time will be given by

$$\zeta(c_1 + \tfrac{1}{2}\omega_\mu) - \zeta(\theta_1 + c_1 + \tfrac{1}{2}\omega_\mu) - e_\mu\,\theta_1 = \frac{\beta_1 h_1}{2 r_1^2}\, t_1. \tag{48}$$

Now if we take $\theta = \theta_1$ and compare (46) and (48), we find

$$\frac{h}{r_0^2}\,t = \frac{h_1}{r_1^2}\,t_1,$$

which by virtue of (47) becomes

$$\frac{t}{t_1} = \frac{r_0^3}{r_1^3} = \frac{R^3}{R_1^3}.$$

We have thus the following analogue of Kepler's Harmonic Law:

Theorem: *Corresponding arcs of similar orbits are described in times which are proportional to the cubes of the middle distances.*[*]

Kepler's Law does not depend upon the eccentricity or shape of the orbit, while the present law is restricted to similar orbits.

Part II.

§ 9. *Classification of Real Orbits.*

It has already been observed that e_λ, e_μ and e_ν are unaltered when b and β are varied in such a manner as to keep their product $b\beta$ constant. If we put $b\beta = \gamma$, then for fixed values of γ different values of b correspond to different starting points on the orbit. Aside from questions of scale and orientation, therefore, we find all the essentially distinct orbits as a one-parameter (γ) set of curves. *Restricting ourselves hereafter to real values of the variables,* these orbits fall naturally into two classes: First, orbits having apses; second, orbits not having apses.

The condition for an apse is

$$\frac{dr}{d\theta} = 0,$$

and from (8) we see that this condition is attained when $\frac{r}{r_0}$ passes through a root of

$$R\left(\frac{r}{r_0}\right) = \tfrac{1}{2}b - \left(\frac{r}{r_0}\right)^2 + \tfrac{1}{2}\beta\left(\frac{r}{r_0}\right)^4 = 0.$$

If, however, these roots are all complex, the condition for an apse can not be satisfied for any real value of $\frac{r}{r_0}$. Considered as a quadratic in $\left(\frac{r}{r_0}\right)^2$, $R\left(\frac{r}{r_0}\right)$ will have complex roots if its discriminant is negative; that is, if

$$1 - b\beta < 0.$$

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[*] Stader gives this result for circular orbits about the origin as center.

From this condition we see that if $b\beta = \gamma$ is less than 1, the orbit will have an apse, and for values of γ greater than 1 there will be no apse. Expressed in terms of γ,

$$\begin{aligned} e_\lambda &= \tfrac{1}{6} + \tfrac{1}{2}\sqrt{1-\gamma}, \\ e_\mu &= -\tfrac{1}{3}, \\ e_\nu &= \tfrac{1}{6} - \tfrac{1}{2}\sqrt{1-\gamma}. \end{aligned} \tag{49}$$

For values of $\gamma < 1$ these expressions are all real. For values of $\gamma > 1$ e_λ and e_μ are conjugate imaginaries.

When e_λ, e_μ and e_ν are all real, it is customary to assign the subscripts in such a manner that

$$e_1 > e_2 > e_3.$$

For all values of γ between $-\infty$ and 0 we see from (49) that

$$e_\lambda > e_\mu > e_\nu,$$

and consequently in this range of γ

$$\lambda = 1, \quad \mu = 2, \quad \nu = 3.$$

When γ lies between 0 and $+1$,

$$e_\lambda > e_\nu > e_\mu,$$

and in this case

$$\lambda = 1, \quad \nu = 2, \quad \mu = 3.$$

When γ is greater than $+1$, e_λ and e_ν are complex, and we follow Weierstrass in taking

$$\lambda = 1, \quad \mu = 2, \quad \nu = 3 \quad \text{(discriminant negative)}.$$

We have therefore three distinct cases depending upon the value of γ. There are also two limiting cases, $\gamma = 0$ and $\gamma = 1$, the solution for which can be expressed by means of trigonometric and logarithmic functions.

§ 10. *Case I.* $-\infty < \gamma < 0$.

We have already mentioned that in this case e_λ, e_μ and e_ν are real and that the subscripts have the order

$$\lambda = 1, \quad \mu = 2, \quad \nu = 3.$$

With these values equations (17) become

$$\begin{aligned} \left(\frac{r}{r_0}\right)^2 &= \frac{2}{\beta}\left[\wp\left(\theta + c + \tfrac{1}{2}\omega_2\right) - e_2\right], \\ \left(\frac{\breve{r}}{\breve{r}_0}\right)^2 = \left(\frac{u}{u_0}\right)^2 &= \frac{2}{b}\left[\wp\left(\theta + c - \tfrac{1}{2}\omega_2\right) - e_2\right]. \end{aligned} \tag{50}$$

This last equation is what we have called the reciprocal solution (see equations (33) *et seq.*)

Since $\gamma = b\beta$ is negative, either β is positive and b negative or the reverse. If β is positive and $c = \omega_1 - \frac{1}{2}\omega_2$, then at $\theta = 0$ the particle is at an apse. This value of c, substituted in (50), determines the corresponding values of b and β to be

$$\tfrac{1}{2}\beta = e_1 - e_2,$$
$$\tfrac{1}{2}b = e_3 - e_2.$$

With these values equations (50) become

$$\left(\frac{r}{r_0}\right)^2 = \frac{1}{e_1 - e_2}\,[\wp(\theta + \omega_1) - e_2],$$
$$\left(\frac{\bar{r}}{\bar{r}_0}\right)^2 = \frac{1}{e_3 - e_2}\,[\wp(\theta + \omega_3) - e_2], \tag{51}$$

and consequently

$$\frac{r}{r_0} = \left[\frac{\wp(\theta) - e_3}{\wp(\theta) - e_1}\right]^{\frac{1}{2}} = \frac{\sigma_3}{\sigma_1}(\theta),$$
$$\frac{\bar{r}}{\bar{r}_0} = \left[\frac{\wp(\theta) - e_1}{\wp(\theta) - e_3}\right]^{\frac{1}{2}} = \frac{\sigma_1}{\sigma_3}(\theta). \tag{52}$$

Since $b = \dfrac{k}{h^2 r_0^2}$ and is negative and h^2 and r_0^2 are certainly positive, k must be negative. That is, the particle is moving under a repulsive force in the first equation of (52). $\beta = \dfrac{k}{h^2 \bar{r}_0^2}$ is positive. Therefore k is positive and the particle is moving under an attractive force in the second equation. If the signs of b and β are interchanged, the value of c at an apse is $c = \omega_1 + \frac{1}{2}\omega_2$, and the solutions (52) are merely interchanged.

The real half period ω_1 and the purely imaginary half period ω_3 are given by the formulæ*

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \qquad \omega_3 = \frac{iK'}{\sqrt{e_1 - e_3}},$$

where

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \qquad K' = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2\phi}},$$

and

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\sqrt{1-\gamma} - 1}{2\sqrt{1-\gamma}},$$
$$k'^2 = \frac{e_1 - e_2}{e_1 - e_3} = \frac{\sqrt{1-\gamma} + 1}{2\sqrt{1-\gamma}}.$$

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* Schwarz, § 27, (3).

As θ increases from 0 to ω_1, $\wp(\theta)$ decreases from $+\infty$ to e_1. Consequently $\frac{r}{r_0}$ increases from 1 to $+\infty$ and $\frac{\bar{r}}{\bar{r}_0}$ decreases from 1 to 0. For the computation of intermediate values the ϑ-functions of Jacobi are convenient, and we find*

$$\begin{aligned} \frac{r}{r_0} &= \frac{\sigma_3}{\sigma_1}(\theta) = \sqrt{\frac{k}{k'}}\,\frac{\vartheta_0}{\vartheta_2}(\nu\,|\,\tau), \\ \frac{\bar{r}}{\bar{r}_0} &= \frac{\sigma_1}{\sigma_3}(\theta) = \sqrt{\frac{k'}{k}}\,\frac{\vartheta_2}{\vartheta_0}(\nu\,|\,\tau), \end{aligned} \tag{53}$$

where $\nu = \frac{\theta}{2\,\omega_1}$ and $\tau = \frac{\omega_3}{\omega_1}$.

The expression for the time (46) for $\frac{r}{r_0}$, as measured from the apse, becomes

$$\tfrac{1}{3}\,\theta + \zeta(\omega_1) - \zeta(\theta + \omega_1) = \tfrac{1}{2}\left(1 + \sqrt{1 - \gamma}\right)\frac{h}{r_0^2}\,t.$$

By differentiating logarithmically the formula†

$$\sigma_\nu(\theta) = e^{-\eta_\nu \theta}\,\frac{\sigma(\theta + \omega_\nu)}{\sigma(\omega_\nu)},$$

where $\eta_\nu = \zeta(\omega_\nu)$, we obtain

$$\frac{\sigma'_\nu}{\sigma_\nu}(\theta) = -\,\eta_\nu + \frac{\sigma'}{\sigma}(\theta + \omega_\nu) = \zeta(\theta + \omega_\nu) - \zeta(\omega_\nu). \tag{54}$$

Putting $\nu = 1$ in this expression, the above expression for the time may be written

$$\tfrac{1}{2}\left(1 + \sqrt{1 - \gamma}\right)\frac{h}{r_0^2}\,t = \tfrac{1}{3}\,\theta - \frac{\sigma'_1}{\sigma_1}(\theta). \tag{55}$$

This expression for t vanishes for $\theta = 0$ and becomes infinite for $\theta = \omega_1$.

The formula for the time for $\frac{\bar{r}}{\bar{r}_0}$ is obtained in the same manner as the above, differing only by the interchange of the subscripts 1 and 3. Making these changes, we get

$$\tfrac{1}{2}\left(\sqrt{1 - \gamma} - 1\right)\frac{h}{\bar{r}_0^2} = \frac{\sigma'_3}{\sigma_3}(\theta) - \tfrac{1}{3}\,\theta. \tag{56}$$

As θ increases from 0 to ω_1, t increases from 0 to $\frac{2\left(\eta_1 - \frac{1}{3}\,\omega_1\right)}{h\left(\sqrt{1 - \gamma} - 1\right)}\,\bar{r}_0^2$, at which time the particle arrives at the origin.

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* Schwarz, § 45, (11), (13).

† Schwarz, § 18, (1).

§ 11. *Case II.* $0 < \gamma < 1$.

In this case $e_\lambda > e_\nu > e_\mu$, and therefore

$$\lambda = 1, \quad \nu = 2, \quad \mu = 3.$$

The derivation of the formulæ is the same as in the former case, the only difference throughout the work being the interchange of the subscripts 2 and 3. We find then

$$\begin{aligned} \frac{r}{r_0} &= \left[\frac{\wp(\theta) - e_2}{\wp(\theta) - e_1}\right]^{\frac{1}{2}} = \frac{\sigma_2}{\sigma_1}(\theta), \\ \frac{\bar{r}}{\bar{r}_0} &= \left[\frac{\wp(\theta) - e_1}{\wp(\theta) - e_2}\right]^{\frac{1}{2}} = \frac{\sigma_1}{\sigma_2}(\theta), \end{aligned} \qquad (57)$$

where

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \qquad \omega_3 = \frac{iK'}{\sqrt{e_1 - e_3}},$$

and

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{1 - \sqrt{1-\gamma}}{1 + \sqrt{1-\gamma}}, \qquad k'^2 = \frac{2\sqrt{1-\gamma}}{1 + \sqrt{1-\gamma}}.$$

In terms of the ϑ functions these solutions are

$$\frac{r}{r_0} = \frac{\sigma_2}{\sigma_1}(\theta) = \sqrt{k}\,\frac{\vartheta_3}{\vartheta_2}(v|\tau). \qquad (58)$$

The expressions for the time are

$$\begin{aligned} &\left(\text{for } \frac{r}{r_0}\right) \quad \tfrac{1}{2}(1 + \sqrt{1-\gamma})\frac{h}{r_0^{\frac{3}{2}}}\,t = \tfrac{1}{3}\theta - \frac{\sigma_1'}{\sigma_1}(\theta), \\ &\left(\text{for } \frac{\bar{r}}{\bar{r}_0}\right) \quad \tfrac{1}{2}(1 - \sqrt{1-\gamma})\frac{h}{\bar{r}_0^{\frac{3}{2}}} \quad = \tfrac{1}{3}\theta - \frac{\sigma_2'}{\sigma_2}(\theta). \end{aligned} \qquad (59)$$

§ 12. *Case III.* $1 < \gamma < +\infty$.

For this range of values of γ we will use the notation

$$\begin{aligned} e_1 &= \tfrac{1}{6} + \tfrac{1}{2}i\sqrt{\gamma - 1}, \\ e_2 &= -\tfrac{1}{3}, \\ e_3 &= \tfrac{1}{6} - \tfrac{1}{2}i\sqrt{\gamma - 1}; \end{aligned}$$

so that $\lambda = 1$, $\mu = 2$, $\nu = 3$ (discriminant negative). With this ordering of the subscripts the half periods ω_1 and ω_3 are conjugate imaginaries, while $\omega_2 = \omega_1 + \omega_3$ is real. For real values of the argument the $\wp$ function varies from $+\infty$ to e_2 as the argument varies from 0 to ω_2.

The solution (17) then is

$$\left(\frac{r}{r_0}\right)^2 = \frac{2}{\beta}\left[\wp\left(\theta + c + \tfrac{1}{2}\omega_2\right) - e_2\right],$$
$$\left(\frac{u}{u_0}\right)^2 = \frac{2}{b}\left[\wp\left(\theta + c - \tfrac{1}{2}\omega_2\right) - e_2\right]. \tag{60}$$

The apses in this case are all imaginary, but the middle point ψ_0 and the zero-point θ_μ are real. If $\theta = 0$ at the middle point, then $c = 0$; and since $\wp\left(\tfrac{1}{2}\omega_2\right) = \wp\left(-\tfrac{1}{2}\omega_2\right)$, we see by comparing the two equations (60) that $b = \beta = \sqrt{\gamma}$, and therefore

$$\left(\frac{r}{r_0}\right)^2 = \frac{2}{\sqrt{\gamma}}\left[\wp\left(\tfrac{1}{2}\omega_2 + \theta\right) - e_2\right],$$
$$\left(\frac{u}{u_0}\right)^2 = \frac{2}{\sqrt{\gamma}}\left[\wp\left(\tfrac{1}{2}\omega_2 - \theta\right) - e_2\right]; \tag{61}$$

therefore

$$\frac{r}{r_0} = \sqrt{\frac{2}{\sqrt{\gamma}}}\,\frac{\sigma_2}{\sigma}\left(\tfrac{1}{2}\omega_2 + \theta\right),$$
$$\frac{u}{u_0} = \sqrt{\frac{2}{\sqrt{\gamma}}}\,\frac{\sigma_2}{\sigma}\left(\tfrac{1}{2}\omega_2 - \theta\right); \tag{62}$$

and comparing these two equations it is evident that

$$\frac{r}{r_0}(\theta) = \frac{1}{\dfrac{r}{r_0}(-\theta)}, \tag{63}$$

or

$$r(\theta)\,.\,r(-\theta) = r_0^2.$$

The formula (43) for the angle between the radius vector and the tangent to the curve becomes in this case

$$\tan^2\phi = \frac{1}{\tfrac{1}{2}\sqrt{\gamma}\left(\dfrac{r_0}{r}\right)^2 - 1 + \tfrac{1}{2}\sqrt{\gamma}\left(\dfrac{r}{r_0}\right)^2}. \tag{64}$$

Since $\left(\dfrac{r}{r_0}\right)$ and its reciprocal enter this expression symmetrically, ϕ has the same value for $+\theta$ as for $-\theta$. It is also readily verified that ϕ is a maximum for $\theta = 0$.

If we put

$$\begin{aligned} e_2 - e_3 &= \rho e^{i\psi}, & \rho &> 0, \\ e_2 - e_1 &= \rho e^{-i\psi}, & 0 &< \psi < \pi, \end{aligned} \tag{65}$$

we find the following value for ω_2:*

$$\omega_2 = \frac{1}{\sqrt{\rho}} K_{(k_1)},$$

where $$k_1^2 = \sin^2 \tfrac{1}{2}\psi.$$

Solving (65) for ρ and ψ, we find

$$\rho = \tfrac{1}{2}\sqrt{\gamma},$$

$$\sin^2 \tfrac{1}{2}\psi = \frac{1 + \sqrt{\gamma}}{2\sqrt{\gamma}}.$$

The formula for the time reduces to

$$\tfrac{1}{2}\sqrt{\gamma}\,\frac{h}{r_0^2}\,t = \tfrac{1}{3}\theta - \zeta(\theta + \tfrac{1}{2}\omega_2) + \zeta(\tfrac{1}{2}\omega_2). \tag{66}$$

§13. *Limiting Cases.*

a) *Limiting Case* $\gamma = 0$.

Since $\gamma = b\beta = 0$, either b or β is zero; and since the solution for either is the reciprocal of the solution for the other, it is immaterial which we choose. We will take $\beta = 0$, and then it follows from their definition (7) that

$$\tfrac{1}{2} b = 1 + \alpha^2 \geqq 1.$$

The differential equation (8) becomes

$$\frac{d\frac{r}{r_0}}{d\theta} = \pm\sqrt{\tfrac{1}{2}b - \left(\frac{r}{r_0}\right)^2}. \tag{67}$$

The solution of this equation is

$$\frac{r}{r_0} = \sqrt{\tfrac{1}{2}b}\cos(\theta + c). \tag{68}$$

If $\theta = 0$ at an apse, we find $c = 0$ and consequently $\tfrac{1}{2}b = 1$, and the solution becomes

$$\frac{r}{r_0} = \cos\theta, \tag{69}$$

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*See Appell et Lacour, "*Théorie des Fonctions Elliptiques*," § 138.

which is a circle passing through the initial point and the origin and having this line as a diameter.

The reciprocal solution is

$$\frac{r}{r_0} = \sec\theta, \tag{70}$$

which is a straight line and implies that the force acting is zero.

b) *Limiting Case* $\gamma = 1$.

Since $b\beta = 1$, $b = \frac{1}{\beta}$ and the differential equation (8) becomes

$$\frac{d\frac{r}{r_0}}{d\theta} = \pm\frac{1}{\sqrt{2b}}\left[\left(\frac{r}{r_0}\right)^2 - b\right]. \tag{71}$$

The solution of this equation is

$$\frac{r}{r_0} = \sqrt{b}\,\frac{1 + e^{\pm\sqrt{2}\theta + c}}{1 - e^{\pm\sqrt{2}\theta + c}}, \tag{72}$$

where, to satisfy the initial conditions, we must take

$$c = \log\frac{1 - \sqrt{b}}{1 + \sqrt{b}}.$$

This solution, together with its reciprocal, represents two spirals asymptotic to the circle $r = r_0\sqrt{b}$, one lying within the circle, the other without the circle. For the special values $b = \beta = 1$ this solution degenerates into the asymptotic circle itself; that is,

$$r = r_0.$$

From the relation $\left(\frac{r}{r_0}\right)^2\frac{d\theta}{dt} = \frac{h}{r_0^2}$ and equation (20) we derive the following expression for the time in terms of θ:

$$\frac{h}{b r_0^2}(t - t_0) = \theta + \sqrt{2}\,\frac{1 - e^{\sqrt{2}\theta + c}}{1 - e^{\sqrt{2}\theta + c}}. \tag{73}$$

§ 14. *Resumé.*

TABLE OF CASES.

γ	e_1	e_2	e_3	ω	k^2
$\gamma < 0$	$\frac{1}{6} + \frac{1}{2}\sqrt{1-\gamma}$	$-\frac{1}{3}$	$\frac{1}{6} - \frac{1}{2}\sqrt{1-\gamma}$	$\omega_1 = \frac{K}{\sqrt{1-\gamma}}$	$\frac{\sqrt{1-\gamma}-1}{2\sqrt{1-\gamma}}$
$\gamma = 0$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\pi/2$	0
$0 < \gamma < 1$	$\frac{1}{6} + \frac{1}{2}\sqrt{1-\gamma}$	$\frac{1}{6} - \frac{1}{2}\sqrt{1-\gamma}$	$-\frac{1}{3}$	$\omega_1 = \frac{2K}{1+\sqrt{1-\gamma}}$	$\frac{1-\sqrt{1-\gamma}}{1+\sqrt{1-\gamma}}$
$\gamma = 1$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$+\infty$	1
$\gamma > 1$	$\frac{1}{6} + \frac{1}{2} i \sqrt{\gamma-1}$	$-\frac{1}{3}$	$\frac{1}{6} - \frac{1}{2} i \sqrt{\gamma-1}$	$\omega_2 = \frac{\sqrt{2}K}{\sqrt[4]{\gamma}}$	$\frac{1+\sqrt{\gamma}}{2\sqrt{\gamma}}$

TABLE OF RESULTS.

γ	$\frac{r}{r_0}$	$\frac{1}{2} b$	$\frac{1}{2}\beta$	t
$\gamma < 0$	(a) $\left[\frac{\wp(\theta)-e_3}{\wp(\theta)-e_1}\right]^{\frac{1}{2}} = \frac{\sigma_3}{\sigma_1}(\theta)$	$(e_3 - e_2)$	$(e_1 - e_2)$	$\frac{r_0^2}{h(e_1-e_2)}\left[\frac{1}{3}\theta - \frac{\sigma_1'}{\sigma_1}(\theta)\right]$
	(b) $\left[\frac{\wp(\theta)-e_1}{\wp(\theta)-e_3}\right]^{\frac{1}{2}} = \frac{\sigma_1}{\sigma_3}(\theta)$	$(e_1 - e_2)$	$(e_3 - e_2)$	$\frac{r_0^2}{h(e_2-e_3)}\left[\frac{\sigma_3'}{\sigma_3}(\theta) - \frac{1}{3}\theta\right]$
$\gamma = 0$	$\cos\theta$	1	0	$\frac{r_0^2}{h}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]$
	$\sec\theta$	0	1	$\frac{r_0^2}{h}\tan\theta$
$0 < \gamma < 1$	(c) $\left[\frac{\wp(\theta)-e_2}{\wp(\theta)-e_1}\right]^{\frac{1}{2}} = \frac{\sigma_2}{\sigma_1}(\theta)$	$(e_2 - e_3)$	$(e_1 - e_3)$	$\frac{r_0^2}{h(e_1-e_3)}\left[\frac{1}{3}\theta - \frac{\sigma_1'}{\sigma_1}(\theta)\right]$
	(d) $\left[\frac{\wp(\theta)-e_1}{\wp(\theta)-e_2}\right]^{\frac{1}{2}} = \frac{\sigma_1}{\sigma_2}(\theta)$	$(e_1 - e_3)$	$(e_2 - e_3)$	$\frac{r_0^2}{h(e_2-e_3)}\left[\frac{1}{3}\theta - \frac{\sigma_2'}{\sigma_2}(\theta)\right]$
$\gamma = 1$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{r_0^2}{h}\ \theta$
$\gamma > 1$	$\frac{\sqrt{2}}{\sqrt[4]{\gamma}}\left[\wp\left(\frac{1}{2}\omega_2+\theta\right)-e_2\right]^{\frac{1}{2}}$ $= \frac{\sqrt{2}}{\sqrt[4]{\gamma}}\frac{\sigma_2}{\sigma}\left(\frac{1}{2}\omega_2+\theta\right)$	$\frac{1}{2}\sqrt{\gamma}$	$\frac{1}{2}\sqrt{\gamma}$	$\frac{2r_0^2}{h\sqrt{\gamma}}\left[\frac{1}{3}\theta - \zeta\left(\frac{1}{2}\omega_2+\theta\right) + \zeta\left(\frac{1}{2}\omega_2\right)\right]$

$$\wp(0) = +\infty, \quad \wp(\omega_1) = e_1, \quad \wp(\omega_2) = e_2, \quad \wp(\omega_3) = e_3.$$

The expressions in this table assume the initial position at an apse or at a middle point.

§15. *Orbit Considered as Function of Parameter γ.*

It is interesting to trace the changes in the form of the orbit as the parameter γ runs through its range of values from $-\infty$ to $+\infty$. For $\gamma = -\infty$, $\omega_1 = 0$ and the orbit is a straight line through the initial point and the origin.

For finite negative values of γ there are two orbits, one lying wholly within the circle, $\frac{r}{r_0} = 1$, the other lying wholly without the same circle (Fig. 1), and so related that the outer orbit is the transform of the inner by reciprocal radii. The inner orbit consists of a series of loops passing through the origin and tangent to the circle, repeating themselves at intervals of $2\,\omega_1$. The curve is closed only when ω_1 is commensurable with π.

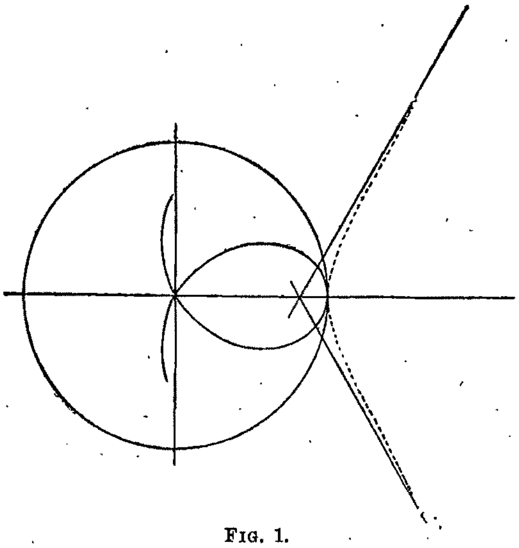

FIG. 1.

$\gamma = -\frac{5}{4}$: $\quad \frac{r}{r_0} = \frac{\sigma_3}{\sigma_1}(\theta), \quad \frac{\bar{r}}{r_0} = \frac{\sigma_1}{\sigma_3}(\theta).$

As γ increases towards 0, ω_1 approaches $\frac{1}{2}\pi$. The loops of the inner orbit broaden out and approach the circle having the initial point and the origin as a diameter. The outer orbit expands very rapidly and approaches the straight line tangent to the circle at the initial point. These limits are attained for $\gamma = 0$ (Fig. 2). These outer orbits are denoted by (a) in the table. It will be observed that the corresponding value of $\frac{1}{2}b$ is $e_3 - e_2$; which is negative. But by definition $\frac{1}{2}b = \frac{k}{2h^2r_0^2}$, which for real orbits can be negative only if k is negative. This means that the outer orbits are described under a repulsive force instead of an attractive force.

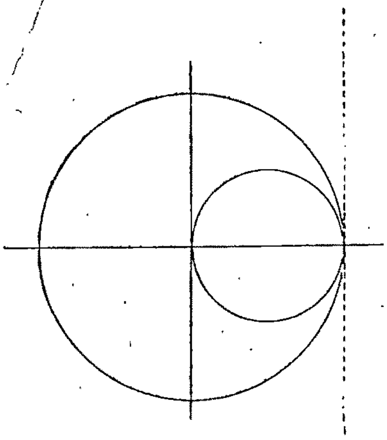

Fig. 2.

$\gamma = 0: \quad \frac{r}{r_0} = \sec\theta, \quad \frac{\bar{r}}{r_0} = \cos\theta.$

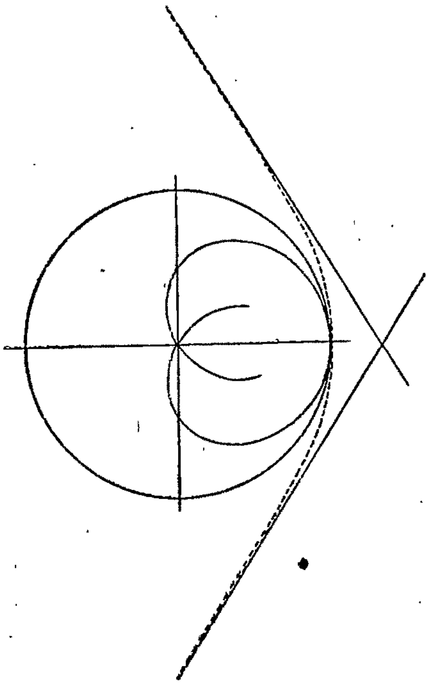

Fig. 3.

$\gamma = \frac{15}{16}: \quad \frac{r}{r_0} = \frac{\sigma_2}{\sigma_1}(\theta), \quad \frac{\bar{r}}{r_0} = \frac{\sigma_1}{\sigma_2}(\theta).$

As γ passes through 0 the value of b for the outer orbits also passes through 0 and becomes positive; the force is therefore changed to attraction. It will be observed that for these orbits the expression for the radius of curvature also changes sign. The orbits therefore curve in the opposite direction. As γ increases from 0 to 1, the curvature increases, ω_1 increases towards $+\infty$, and as $\gamma \doteq 1$ the orbit approaches the hyperbolic spiral (equation (72)) asymptotic to the circle. The loops of the inner orbit continue to widen out (Fig. 3), and as $\gamma \doteq 1$ the inner orbit also approaches the hyperbolic spiral asymptotic to the circle on the inside. These limiting spirals are actually attained for the value $\gamma = 1$ if the initial position is not taken at an apse. Otherwise the limiting orbit is the asymptotic circle itself (Fig. 4). In order to avoid confusion in the figure only that branch of the outer curve has been drawn corresponding to positive values of θ, and of the inner orbit the branch corresponding to negative values. As γ passes through the value 1 these branches which are asymptotic to the circle

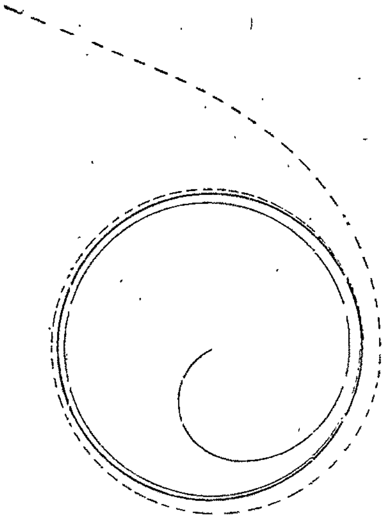

FIG. 4.

$\gamma = 1.$

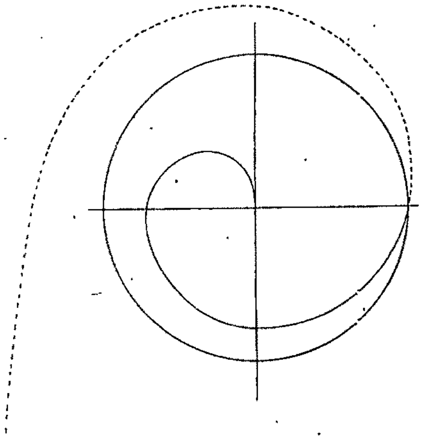

FIG. 5.

$$\gamma = \frac{201}{200}: \qquad \frac{r}{r_0} = \sqrt{\frac{2}{\sqrt{\gamma}}}\,\frac{\sigma_2}{\sigma}\,(\tfrac{1}{2}\omega_2 + \theta).$$

unite, and for values of $\gamma > 1$ the orbit crosses the circle and the apse ceases to exist. The point of crossing becomes a middle point, and the reciprocal relation between the inner portion and the outer with respect to this point continues as before. For values of γ but little greater than 1, the period ω_2 is very great, but decreases as γ increases. The curve which now passes through the origin and infinity straightens out as γ increases (Fig. 6). As $\gamma \doteq +\infty$, ω_2 decreases and approaches the value 0, and the orbit again approaches the straight line.

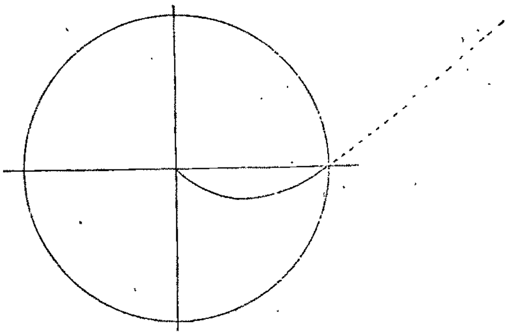

Fig. 6.

$$\gamma = 16: \qquad \frac{r}{r_0} = \sqrt{\frac{2}{\sqrt{\gamma}}}\, \frac{\sigma_2}{\sigma}\left(\tfrac{1}{2}\omega_2 + \theta\right).$$

The University of Chicago, *June* 27, 1907.

On a Group of Transformations which Occurs in the Problem of Several Bodies.

By Edgar Odell Lovett.

Given a system of $n+1$ bodies consisting of a fixed body $(0, 0, 0; \mu)$ and n others $(x_i, y_i, z_i; m_i)$, mutually attracting one another by central forces varying directly as the masses and as any arbitrary function of the distance; to determine the motion of the n bodies about the fixed center we arrive at a system of $6n$ differential equations of the first order in the canonical form:

$$\left.\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial F}{\partial \xi_i}, & \frac{dy_i}{dt} &= -\frac{\partial F}{\partial \eta_i}, & \frac{dz_i}{dt} &= -\frac{\partial F}{\partial \zeta_i}, \\ \frac{d\xi_i}{dt} &= \frac{\partial F}{\partial x_i}, & \frac{d\eta_i}{dt} &= \frac{\partial F}{\partial y_i}, & \frac{d\zeta_i}{dt} &= \frac{\partial F}{\partial z_i}, \end{aligned}\right\} \quad (i = 1, 2, \ldots, n), \tag{1}$$

where ξ_i, η_i, ζ_i are proportional to the projections of the velocities of the bodies on the axes of coordinates, and the function F is of the form

$$F = U - \sum_{i=1}^{n} \frac{\xi_i^2 + \eta_i^2 + \zeta_i^2}{2 m_i}, \tag{2}$$

the force-function being designated by U.

Let new variables

$$\left.\begin{aligned} q_{ij} &= x_i x_j + y_i y_j + z_i z_j, & q_{ji} &= q_{ij}, \\ r_{ij} &= \xi_i \xi_j + \eta_i \eta_j + \zeta_i \zeta_j, & r_{ji} &= r_{ij}, \\ s_{ij} &= x_i \xi_j + y_i \eta_j + z_i \zeta_j, & s_{ji} &\neq s_{ij}, \end{aligned}\right\} \quad (i, j = 1, 2, 3, \ldots, n), \tag{3}$$

be introduced. These variables are of the same form as those employed by Bertrand* in the problem of three bodies. They are $n(2n+1)$ in number and are not all distinct. In fact we readily see from their form that the symmetrical determinant

$$\Delta = \begin{vmatrix} q_{ij} & s_{ij} \\ s_{ij} & r_{ij} \end{vmatrix}, \quad (i, j = 1, 2, \ldots, n), \tag{4}$$

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*Bertrand, Mémoire sur l'intégration des équations différentielles de la mécanique, *Journal de Liouville*, Ser. I, Vol. XVII (1852), pp. 393–436.

where q_{ij} represents the square array of n^2 elements obtained by giving to i, j the values $1, 2, \ldots, n$, and all its minors down to and including all the $\frac{1}{2}\binom{2n}{4}\left\{\binom{2n}{4}+1\right\}$ which are determinants of the fourth order vanish, and that no one of the $\frac{1}{2}\binom{2n}{3}\left\{\binom{2n}{3}+1\right\}$ which are of the third order vanishes. These $\frac{1}{2}\binom{2n}{4}\left\{\binom{2n}{4}+1\right\}$ conditions among $n(2n+1)$ quantities are far too numerous; they can be reduced to proper bounds by means of a theorem of Kronecker.* We find in fact that the vanishing of all the $\frac{1}{2}\binom{2n}{4}\left\{\binom{2n}{4}+1\right\}$ fourth order subdeterminants of the above symmetrical determinant is a consequence of the vanishing of $(n-1)(2n-3)$ properly chosen independent fourth order subdeterminants, and this choice can be made in $\frac{1}{2}\binom{2n}{3}\left\{\binom{2n}{3}+1\right\}$ ways. Then by the aid of these independent relations $(n-1)(2n-3)$ of the variables (3) can be eliminated if they be employed in problem (1); there would remain $6n-3$ independent variables, which would be sufficient since a loss of three from the original $6n$ independent variables can be accounted for by a change in orientation. On making $n=2$ in Δ we have Bour's determinant† the vanishing of which expresses the single relation among Bertrand's ten variables (3) in the problem of three bodies.

In the variables (3) the force-function U becomes

$$U=\sum_{i=1}^{n}\mu m_i f(\sqrt{q_{ii}})-\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j f(\sqrt{q_{ii}+q_{jj}-2q_{ij}})\,; \tag{5}$$

accordingly the partial derivatives of F are of the form

$$\frac{\partial F}{\partial x_i}=\mu_i x_i+\sum_{j=1}^{n}\mu_{ij}x_j\,,\quad \frac{\partial F}{\partial \xi_i}=-\frac{\xi_i}{m_i}\,, \tag{6}$$

where the quantities

$$\left.\begin{aligned}\mu_i&=\mu m_i\,\frac{f'(\sqrt{q_{ii}})}{\sqrt{q_{ii}}}-\sum_{j=1}^{n}\mu_{ij}\,,\\ \mu_{ij}&=m_i m_j\,\frac{f'(\sqrt{q_{ii}+q_{jj}-2q_{ij}})}{\sqrt{q_{ii}+q_{jj}-2q_{ij}}}=\mu_{ji}\end{aligned}\right\} \tag{7}$$

are coefficients depending on the forces and expressed in terms of the q's alone.

Then in virtue of (1) the variables (3) satisfy the following system of ordinary differential equations:

$$\left.\begin{aligned}\frac{dq_{ij}}{dt}&=\frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}\,,\\ \frac{dr_{ij}}{dt}&=\mu_i s_{ij}+\mu_j s_{ji}+\mu_{ij}(s_{ii}+s_{jj})+\sum_{k=1}^{n}\mu_{jk}s_{ki}+\sum_{l=1}^{n}\mu_{il}s_{lj}\,,\\ \frac{ds_{ij}}{dt}&=\mu_j q_{ij}+\mu_{ij}q_{ii}+\frac{r_{ij}}{m_i}+\sum_{k=1}^{n}\mu_{jk}q_{ik}\,;\end{aligned}\right\}\ (i, j=1, 2, \ldots, n), \tag{8}$$

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[*] Kronecker, Bemerkungen zur Determinanten-Theorie, *Crelle's Journal*, Vol. LXXII (1870), pp. 152–175.

[†] Bour, Mémoire sur le problème des trois corps, *Journal de l'École Polytechnique*, Vol. XXI (1856), pp. 35–58.

these equations are the generalizations of Bour's equations in the problem of three bodies.

If now, in this problem of the motion of n bodies about a fixed center under forces varying as an arbitrary function of the distance as formulated above, we seek those integrals which do not involve the law of force, we have to find those functions ϕ of all the q's, r's and s's not containing the μ's, whose total derivative with regard to the time

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial\phi}{\partial q_{ij}}\frac{dq_{ij}}{dt}+\frac{\partial\phi}{\partial r_{ij}}\frac{dr_{ij}}{dt}+\frac{\partial\phi}{\partial s_{ij}}\frac{ds_{ij}}{dt}\right\} \tag{9}$$

vanishes independently of the μ's when the total derivatives are replaced by their values (8).

From the absolute term of the equation thus formed we have the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left(\frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}\right)\phi_{q_{ij}}+\frac{r_{ij}}{m_i}\phi_{s_{ij}}\right\}=0; \tag{10}$$

from the coefficients of the μ_i the following n equations:

$$b_i \equiv 2w_i\phi_{v_i}+u_i\phi_{w_i}+\sum_{j=1}^{n}(s_{ij}\phi_{r_{ij}}+q_{ij}\phi_{s_{ji}})=0, \quad (i=1,2,\ldots,n); \tag{11}$$

and finally from the terms in which the μ_{ij} appear the following $\frac{1}{2}n(n-1)$ equations:

$$\left.\begin{aligned} d_{ij}\equiv d_{ji}\equiv 2s_{ji}\phi_{v_i}+2s_{ij}\phi_{v_j}+q_{ij}(\phi_{w_i}+\phi_{w_j})+(w_i+w_j)\phi_{r_{ij}}+u_i\phi_{s_{ij}}+u_j\phi_{s_{ji}}\\ +\sum_{k=1}^{n}(s_{ik}\phi_{r_{jk}}+s_{jk}\phi_{r_{ik}}+q_{jk}\phi_{s_{ki}}+q_{ki}\phi_{s_{kj}})=0,\ (i,j=1,2,\ldots,n), \end{aligned}\right\} \tag{12}$$

where for brevity we have put

$$q_{ii}=u_i, \qquad r_{ii}=v_i, \qquad s_{ii}=w_i. \tag{13}$$

Combining these $\frac{1}{2}n(n+1)+1$ equations (10), (11), (12) in all possible pairs, by Poisson's operation, we obtain the following complete system of $n(2n+1)$ linear partial differential equations of the first order:

$$\left.\begin{aligned} &a_i\equiv 2w_i\phi_{u_i}+v_i\phi_{w_i}+\sum_{j=1}^{n}(s_{ji}\phi_{q_{ij}}+r_{ij}\phi_{s_{ij}})=0; \quad b_i=0;\\ &c_i\equiv 2u_i\phi_{u_i}-2v_i\phi_{v_i}+\sum_{j=1}^{n}(q_{ij}\phi_{q_{ij}}-r_{ij}\phi_{r_{ij}}+s_{ij}\phi_{s_{ij}}-s_{ji}\phi_{s_{ji}})=0; \quad d_{ij}=0;\\ &e_{ij}\equiv 2q_{ij}\phi_{u_i}-2r_{ij}\phi_{v_j}+s_{ji}(\phi_{w_i}-\phi_{w_j})+u_j\phi_{q_{ij}}-v_i\phi_{r_{ij}}+(w_j-w_i)\phi_{s_{ij}}\\ &\qquad+\sum_{k=1}^{n}(q_{jk}\phi_{q_{ik}}-r_{ki}\phi_{r_{jk}}+s_{jk}\phi_{s_{ik}}-s_{ki}\phi_{s_{kj}})=0;\\ &f_{ij}\equiv 2s_{ij}\phi_{u_i}+2s_{ji}\phi_{u_j}+r_{ij}(\phi_{w_i}+\phi_{w_j})+(w_i+w_j)\phi_{r_{ij}}+v_j\phi_{s_{ij}}+v_i\phi_{s_{ji}}\\ &\qquad+\sum_{k=1}^{n}(s_{kj}\phi_{q_{ik}}+s_{ki}\phi_{q_{jk}}+r_{jk}\phi_{s_{ik}}+r_{ki}\phi_{s_{jk}})=0;\\ &\qquad d_{ji}=d_{ij}, \qquad e_{ji}\ddagger e_{ij}, \qquad f_{ji}=f_{ij}, \qquad (i,j=1,2,\ldots,n). \end{aligned}\right\} \tag{14}$$

These equations are the generalizations of those given by Gravé[*] for the case $n = 2$.

The preceding remarks are taken from a previous paper,[†] in which was noted incidentally the fact that the $n(2n+1)$ operators

$$a_i,\ b_i,\ c_i,\ d_{ij},\ e_{ij},\ f_{ij} \tag{15}$$

constitute a continuous group of transformations in Lie's sense. That these infinitesimal transformations generate a group was pointed out in an unpublished paper read by the writer before the American Mathematical Society, December 29, 1902.

It is the object of the present note to construct the invariants of this group by the methods of Sophus Lie; the variables will be regarded as independent, but for convenience the notation will not be changed.

The group property of the family (15) is exhibited by the following table of values of the symbol of Poisson:

$$\left.\begin{array}{llll}
(a_i,\ b_i) = -c_i; & (a_i, c_i) = 2a_i; & (a_i, d_{ij}) = -e_{ij}; & (a_i, e_{ji}) = f_{ij}; \\
(b_i,\ c_i) = -2b_i; & (b_i, e_{ij}) = -d_{ij}; & (b_i, f_{ij}) = e_{ji}; & \\
(c_i,\ d_{ij}) = d_{ij}; & (c_i, e_{ij}) = -(c_j, e_{ij}) = -e_{ij}; & (c_i, e_{ji}) = -(c_j, e_{ji}) = e_{ji}; & \\
(c_i,\ f_{ij}) = -f_{ij}; & & & \\
(d_{ij}, e_{ij}) = -2b_j; & (d_{ij}, e_{jk}) = -d_{ki}; & (d_{ij}, f_{ij}) = c_i + c_j; & \\
(d_{ij}, f_{jk}) = e_{kj}; & & & \\
(e_{ij},\ e_{jk}) = -e_{ki}; & (e_{ij}, e_{ki}) = e_{kj}; & (e_{ij},\ e_{ji}) = c_j - c_i; & \\
(e_{ij},\ f_{ij}) = -2a_i; & (e_{ij}, f_{jk}) = -f_{ki}; & &
\end{array}\right\} \tag{16}$$

from which have been omitted both those which are identically zero and those which are the simple inverses of those given.

To find the invariants of the group we have only to integrate the complete system

$$a_i = b_i = c_i = d_{ij} = e_{ij} = f_{ij} = 0, \qquad (i, j = 1, 2, \ldots, n), \tag{17}$$

of linear partial differential equations of the first order.

This system (17) consists of $n(2n+1)$ equations in as many variables; hence that a solution exist it is necessary that the determinant of the coefficients of the partial derivatives should vanish. This determinant of the $n(2n+1)$th order refuses to yield to the ordinary methods of evaluation. The question of the vanishing of the determinant, however, takes care of itself, for after a few reductions of the order of the system we shall find the condition of integrability satisfied. Moreover we may convince ourselves of the integrability of the system by remarking that the determinant Δ is a solution of the system (17), as may be verified immediately by the aid of the fundamental theorems in the expansion of determinants. Furthermore it is easy to convince one's self that there are not more than n solutions. In fact, writing down the determinant of the coefficients of the partial derivatives of the system (17), it appears that there is a determinant of order $2n^2$ in it whose principal diagonal is

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[*] Gravé, Sur le problème des trois corps, *Nouvelles Annales de Mathématiques*, Ser. III, Vol. XV (1896), pp. 537–547.

[†] Lovett, On a problem including that of several bodies and admitting of an additional integral, *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 491–495.

$$w_1^2 w_2^2 \ldots . w_n^2 \prod_{ij} (w_i^2 - w_j^2)^2 \tag{18}$$

and unique. The existence of this unique term can not be used to prove the non-vanishing of a subdeterminant of a higher order, since its coefficient in the original determinant is a determinant all of whose elements are zero. We infer then that the system of the $n(2n+1)$th order has at most n solutions.

Let us now apply the method of Boole and Korkine to the reduction of the order of the system in hand.

The equation

$$a_1 \equiv 2 w_1 \phi_{u_i} + v_1 \phi_{w_i} + \sum_{1}^{n} (s_{ji} \phi_{q_{ij}} + r_{ij} \phi_{s_{ij}}) = 0, \qquad (j \mp 1), \tag{19}$$

is equivalent to the simultaneous system of total differential equations

$$\frac{du_1}{2w_1} = \frac{dw_1}{v_1} = \frac{dq_{12}}{s_{21}} = \frac{dq_{13}}{s_{31}} = \ldots . = \frac{dq_{1n}}{s_{n1}} = \frac{ds_{12}}{r_{12}} = \frac{ds_{13}}{r_{13}} = \ldots . = \frac{ds_{1n}}{r_{1n}}, \tag{20}$$

of which we have the following $2n-1$ independent integrals:

$$\left.\begin{array}{c} u_1 v_1 - w_1^2 = \xi_1, \quad w_1 r_{1i} - v_1 s_{1i} = \xi_{1i}, \quad s_{1i} s_{i1} - q_{1i} r_{1i} = \eta_{1i}, \\ \xi_{1i} \mp \xi_{i1}, \quad \eta_{1i} = \eta_{i1}, \quad (i = 2, 3, \ldots ., n). \end{array}\right\} \tag{21}$$

Introducing the quantities (21) as new variables into the remaining equations of the system (17) with a view to eliminating the $2n$ old variables

$$u_1, \; w_1, \; q_{12}, \; q_{13}, \ldots ., \; q_{1n}, \; s_{12}, \; s_{13}, \ldots ., \; s_{1n}, \tag{22}$$

there results a system of $(n+1)(2n-1)$ equations in which the variables (22) do not appear explicitly. Indicating by an upper index unity the result of the substitution (21) the first member of the last-mentioned system is the equation

$$a_2^1 = 0, \tag{23}$$

which is equivalent to the following simultaneous system:

$$\frac{du_2}{2\,w_2}=\frac{dw_2}{v_2}=\frac{dq_{23}}{s_{32}}=\frac{dq_{24}}{s_{42}}=\ldots\ldots=\frac{dq_{2n}}{s_{n2}}=\frac{ds_{21}}{r_{12}}=\frac{ds_{23}}{r_{23}}=\frac{ds_{24}}{r_{24}}=\ldots\ldots=\frac{ds_{2n}}{r_{2n}}, \quad (24)$$

of which we have the $2\,(n-1)$ independent algebraic integrals

$$\left.\begin{array}{l} u_2\,v_2-w_2^2=\xi_2, \quad w_2\,r_{i2}-v_2\,s_{2i}=\xi_{2i}, \quad s_{2j}\,s_{j2}-q_{2j}\,r_{2j}=\eta_{2j}, \\ \xi_{i2}\ddagger\xi_{2i}, \quad \eta_{j2}=\eta_{2j}, \quad (i=1,3,4,\ldots,n;\; j=3,4,\ldots,n). \end{array}\right\} \quad (25)$$

If the latter be employed as new variables to eliminate the $2\,n-1$ old ones,

$$u_2, \quad w_2, \quad q_{2i}, \quad s_{21}, \quad s_{2i}, \quad (i=3,4,\ldots,n), \quad (26)$$

there results a system of $2\,n^2+n-2$ equations whose inital member is

$$a_3^{12}=0. \quad (27)$$

Repeating this process $n-1$ times we arrive at the system whose first member is

$$a_n^{123\ldots.n-1}=0, \quad (28)$$

the corresponding total differential system possessing the following n independent algebraic integrals:

$$u_n\,v_n-w_n^2=\xi_n, \quad w_n\,r_{ni}-v_n\,s_{ni}=\xi_{ni}, \quad (i=1,\,2,\ldots,\,n-1). \quad (29)$$

Up to this point all of the variables

$$u_i,\; w_i,\; q_{ij},\; s_{ij},\; s_{ji}, \quad (i,j=1,\,2,\ldots,\,n), \quad (30)$$

have been eliminated except

$$u_n,\; w_n,\; s_{n1},\; s_{n2},\ldots,\; s_{n\,n-1}. \quad (31)$$

On introducing the variables (29) to eliminate the variables (31) the original system (17) is reduced after this the nth substitution to the following system of $2n^2$ equations:

$$c_i^{12\ldots.n}\equiv 2v_i\,\phi_{v_i}+\sum_{j=1}^{n}(r_{ij}\,\phi_{r_{ij}}+\xi_{ij}\phi_{\xi_{ij}}+\xi_{ji}\,\phi_{\xi_{ji}})=0, \quad (i=1,\,2,\ldots,n;\, j\ddagger i); \quad (32)$$

$$\begin{aligned} f_{ij}^{12\ldots.n}\equiv & -2\xi_{ij}\;\phi_{\xi_i}-2\xi_{ji}\,\phi_{\xi_j}+(r_{ij}^2-v_i\,v_j)\,(\phi_{\xi_{ij}}+\phi_{\xi_{ji}})-(\xi_{ij}+\xi_{ji})\,\phi_{\eta_{ij}} \\ & +\sum_1^{n}{}^{k}\Big\{(r_{ij}\,r_{ki}-v_i\,r_{jk})\,\phi_{\xi_{ik}}+(r_{ij}r_{jk}-v_j\,r_{ik})\,\phi_{\xi_{jk}} \\ & +\frac{1}{v_k}(r_{jk}\,\xi_{ki}-r_{ki}\,\xi_{kj})\,(\phi_{\eta_{jk}}-\phi_{\eta_{ki}})\Big\}=0, \\ & \qquad (i,j=1,\;2,\ldots,\;n;\; k\ddagger i,\; k\ddagger j); \end{aligned} \quad (33)$$

$$\begin{aligned} (v_i\,b_i^{12\ldots.i}+w_i\,c_i^{12\ldots.i})^{i+1,\,i+2,\ldots,\,n}\equiv & -\sum_1^{n}{}^{j}\Big\{\xi_{ij}\,\phi_{r_{ij}}+r_{ij}\,\xi_i\,\phi_{\xi_{ij}} \\ & +\frac{1}{r_{ij}}(v_i\,v_j\,\eta_{ij}-\xi_{ij}\,\xi_{ji})\;\phi_{\xi_{ji}}\Big\}=0, \qquad (i=1,\,2,\ldots,\,n); \end{aligned} \quad (34)$$

$$\begin{aligned}(w_i f_{ij}^{12\ldots i} - v_i e_{ji}^{12\ldots i})^{i+1,\, i+2,\ldots,\, n} \equiv\ & 2v_i\, r_{ij} \Phi_{v_i} + v_i\, v_j\, \Phi_{r_{ij}} + \sum_1^n{}^k v_i\, r_{jk}\, \Phi_{r_{ki}} \\ & + 2r_{ij}\, \xi_i\, \Phi_{\xi_i} + \frac{2}{r_{ij}} (v_i\, v_j\, \eta_{ij} - \xi_{ij}\, \xi_{ji})\, \Phi_{\xi_j} + r_{ij}\, \xi_{ij}\, (\Phi_{\xi_{ij}} + \Phi_{\xi_{ji}}) \\ & + \sum_1^n{}^k \{ v_i\, \xi_{kj}\, \Phi_{\xi_{ki}} + (r_{jk}\, \xi_{ij} - v_j\, \xi_{ik})\, \Phi_{\xi_{jk}} + (2r_{ij}\, \xi_{ik} - r_{ki}\, \xi_{ij})\, \Phi_{\xi_{ik}} \} \\ & + \left[r_{ij}\, \xi_i + \frac{1}{r_{ij}} (v_i\, v_j\, \eta_{ij} - \xi_{ij}\, \xi_{ji}) \right] \Phi_{\eta_{ij}} + \sum_1^n{}^k \left[\frac{r_{jk}}{v_k r_{ki}} (\xi_{ik}\, \xi_{ki} - v_k\, v_i\, \eta_{ki}) \right. \\ & \left. - \frac{\xi_{ik}\, \xi_{kj}}{v_k} \right] (\Phi_{\eta_{jk}} - \Phi_{\eta_{ki}}) = 0, \quad (i, j = 1, 2, \ldots, n;\ k \neq i,\ k \neq j); \quad (35)\end{aligned}$$

$$\begin{aligned}& [w_j\, (w_i\, f_{ij}^{12\ldots i} - v_i\, e_{ji}^{12\ldots i})^{i+1,\, i+2,\ldots,\, n-1} - v_j\, (v_i\, d_{ij}^{12\ldots i} + w_i\, e_{ij}^{12\ldots i})^{i+1,\, i+2,\ldots,\, n-1}]^n \\ & \equiv 2v_i\ \xi_{ji}\ \Phi_{v_i} + 2v_j\, \xi_{ij}\, \Phi_{v_j} + \sum_1^n{}^k (v_i\, \xi_{jk}\, \Phi_{r_{ki}} + v_j\, \xi_{ik}\, \Phi_{r_{jk}}) + 2\xi_i\, \xi_{ji}\, \Phi_{\xi_i} + 2\xi_j\, \xi_{ij}\, \Phi_{\xi_j} \\ & \quad + [v_i\, v_j\, (\xi_i + \eta_{ij}) + \xi_{ij}\, \xi_{ji}]\, \Phi_{\xi_{ij}} + [v_i\, v_j\, (\xi_j + \eta_{ij}) + \xi_{ij}\, \xi_{ji}]\, \Phi_{\xi_{ji}} \\ & \quad + \sum_1^n{}^k \left\{ \frac{v_i}{r_{jk}} (\xi_{jk}\, \xi_{kj} - v_j\, v_k\, \eta_{jk})\, \Phi_{\xi_{ki}} + \frac{v_j}{r_{ki}} (\xi_{ki}\, \xi_{ik} - v_k\, v_i\, r_{ki})\, \Phi_{\xi_{kj}} + [2\xi_{ik}\, \xi_{ji} \right. \\ & \quad \left. + \frac{r_{ki}}{r_{ij}} (v_i\, v_j\, \eta_{ij} - \xi_{ij}\, \xi_{ji})]\, \Phi_{\xi_{ik}} + [2\xi_{jk}\, \xi_{ij} + \frac{r_{jk}}{r_{ij}} (v_i\, v_j\, \eta_{ij} - \xi_{ij}\, \xi_{ji})\, \Phi_{\xi_{jk}} \right\} \\ & \quad + (\xi_i\, \xi_{ji} + \xi_j\, \xi_{ij})\, \Phi_{\eta_{ij}} + \sum_1^n{}^k \left\{ \frac{\xi_{jk}}{v_k\, r_{ki}} (v_k\, v_i\, \eta_{ki} - \xi_{ki}\, \xi_{ik}) - \frac{\xi_{ki}}{v_k\, r_{jk}} (v_j\, v_k\, \eta_{jk} \right. \\ & \quad \left. - \xi_{jk}\, \xi_{kj}) \right\} (\Phi_{\eta_{ik}} - \Phi_{\eta_{kj}}) = 0, \qquad (i, j = 1, 2, \ldots, n;\ k \neq i,\ k \neq j); \qquad (36)\end{aligned}$$

equations whose construction has been facilitated by relations such as the following:

$$\left.\begin{aligned} w_i\, s_{ji} - q_{ij}\, v_i &= \frac{s_{ji}\, \xi_{ij} + v_i\, \eta_{ij}}{r_{ij}}, \\ r_{ik}\, s_{kj} - r_{jk}\, s_{ki} &= \frac{r_{jk}\, \xi_{ki} - r_{ki}\, \xi_{kj}}{v_k}, \\ s_{ij}\, s_{ki} - q_{ki}\, r_{ij} &= \frac{r_{ij}}{v_i\, r_{ki}} (v_i\, \eta_{ki} + s_{ki}\, \xi_{ik}) - \frac{s_{ki}\, \xi_{ij}}{v_i}, \\ q_{ij}\, s_{ki} - q_{ki}\, s_{ji} &= \frac{s_{ji}\, s_{ki}}{v_i\, r_{ij}\, r_{ki}} (r_{ij}\, \xi_{ik} - r_{ki}\, \xi_{ij}) + \frac{s_{ji}\, \eta_{ki}}{r_{ki}} - \frac{s_{ki}\, \eta_{ij}}{r_{ij}}, \\ & (i, j, k = 1, 2, 3, \ldots, n), \end{aligned}\right\} \quad (37)$$

where

$$\xi_i = u_i v_i - w_i^2, \quad \xi_{ij} = w_i\, r_{ij} - v_i\, s_{ij} \neq \xi_{ji}, \quad \eta_{ij} = s_{ij}\, s_{ji} - q_{ij}\, r_{ij} = \eta_{ji}, \quad (i, j = 1, 2, \ldots, n). \qquad (38)$$

The equations (32) themselves constitute a complete system of n equations in $\frac{1}{2}n(3n-1)$ partial derivatives; the system then possesses $\frac{3}{2}n(n-1)$ independent solutions and these are readily found to be

$$\left.\begin{aligned} &\lambda_{ij}=\rho_{ij}\,\rho_{ji}=\frac{v_i}{r_{ij}}\,\frac{v_j}{r_{ji}}=\frac{v_i\,v_j}{r_{ij}^2}=\lambda_{ji}, \\ &\beta_{ij}=\frac{\xi_{ij}}{r_{ij}},\quad \beta_{ji}=\frac{\xi_{ji}}{r_{ij}},\quad \beta_{ji}\neq\beta_{ij},\qquad (i,j=1,\,2,\ldots,n). \end{aligned}\right\}\tag{39}$$

Introducing these variables into the remaining equations (33), (34), (35), (36), eliminating the $\frac{1}{2}n(3n-1)$ old variables

$$v_1,\ v_2,\ldots,\ v_n,\ r_{ij},\ \xi_{ij},\ \xi_{ji},\qquad (i,j=1,\,2,\ldots,\,n),\tag{40}$$

we obtain the following system:

$$(34)^{\lambda}\equiv\sum_{1}^{n}{}_{j}\{2\lambda_{ij}\,\beta_{ij}\,\phi_{\lambda_{ij}}+(\xi_i+\beta_{ij}^2)\,\phi_{\beta_{ij}}+\lambda_{ij}\,\eta_{ij}\,\phi_{\beta_{ji}}\}=0,$$
$$(i=1,\,2,\ldots,\,n);\tag{41}$$

$$(33)^{\lambda}\equiv-2\beta_{ij}\,\phi_{\xi_i}-2\beta_{ji}\,\phi_{\xi_j}-(\beta_{ij}+\beta_{ji})\,\phi_{\eta_{ij}}+\sum_{1}^{n}{}_{k}\frac{\lambda_{ij}}{\rho_{ijk}}(\beta_{ki}-\beta_{kj})\,(\phi_{\eta_{jk}}-\phi_{\eta_{ki}})$$
$$+(1-\lambda_{ij})\,(\phi_{\beta_{ij}}+\phi_{\beta_{ji}})+\sum_{1}^{n}{}_{k}\left\{\left(1-\frac{\rho_{ijk}}{\lambda_{jk}}\right)\phi_{\beta_{ik}}+\left(1-\frac{\rho_{ijk}}{\lambda_{ki}}\right)\phi_{\beta_{jk}}\right\}=0,$$
$$(i,j,\,k=1,\,2,\ldots,\,n;\ k\neq i,\,k\neq j);\tag{42}$$

$$(35)^{\lambda}\equiv 2\xi_i\,\phi_{\xi_i}+2(\lambda_{ij}\,\eta_{ij}-\beta_{ij}\,\beta_{ji})\,\phi_{\xi_j}+(\xi_i+\lambda_{ij}\,\eta_{ij}-\beta_{ij}\,\beta_{ji})\,\phi_{\eta_{ij}}$$
$$+\sum_{1}^{n}{}_{k}\left\{\frac{\lambda_{ij}}{\rho_{ijk}}(\beta_{ik}\,\beta_{ki}-\lambda_{ki}\,\eta_{ki}-\beta_{ik}\,\beta_{kj})\,(\phi_{\eta_{jk}}-\phi_{\eta_{ki}})\right\}+2\lambda_{ij}\,(1-\lambda_{ij})\,\phi_{\lambda_{ij}}$$
$$+2\sum_{1}^{n}{}_{k}\lambda_{ki}\left(1-\frac{\rho_{ijk}}{\lambda_{jk}}\right)\phi_{\lambda_{ki}}+\beta_{ij}\,(1-\lambda_{ij})\,\phi_{\beta_{ij}}+(\beta_{ij}-\lambda_{ij}\,\beta_{ji})\,\phi_{\beta_{ji}}$$
$$+\sum_{1}^{n}{}_{k}\left\{\frac{\rho_{ijk}}{\lambda_{jk}}[(\beta_{kj}-\beta_{ki})\,\phi_{\beta_{ki}}-\beta_{ik}\,\phi_{\beta_{ik}}]+(2\beta_{ik}-\beta_{ij})\,\phi_{\beta_{ik}}\right.$$
$$\left.+\left(\beta_{ij}-\frac{\rho_{ijk}}{\lambda_{ki}}\beta_{ik}\right)\phi_{\beta_{jk}}\right\}=0,\qquad (i,j,\,k=1,\,2,\ldots,\,n;\ k\neq i,\,k\neq j);\tag{43}$$

$$
\begin{aligned}
(36)^{\lambda} \equiv\; & 2\xi_i\,\beta_{ji}\,\phi_{\xi_i} + 2\xi_j\,\beta_{ij}\,\phi_{\xi_j} + (\xi_i\,\beta_{ji} + \xi_j\,\beta_{ij})\,\phi_{\eta_{ij}} \\
& + \sum_1^n{}^k \frac{\lambda_{ij}}{\rho_{ijk}}\left[\beta_{jk}\,(\lambda_{ki}\,\eta_{ki} - \beta_{ki}\,\beta_{ik}) - \beta_{ik}\,(\lambda_{jk}\,\eta_{jk} - \beta_{jk}\,\beta_{kj})\right](\phi_{\eta_{ki}} - \phi_{\eta_{jk}}) \\
& + 2\lambda_{ij}\,(\beta_{ij} + \beta_{ji})\,\phi_{\lambda_{ij}} + 2\sum_1^n{}^k \left[\lambda_{jk}\left(\beta_{ij} - \frac{\rho_{ijk}}{\lambda_{ki}}\beta_{ik}\right)\phi_{\lambda_{jk}} + \lambda_{ki}\left(\beta_{ji} - \frac{\rho_{ijk}}{\lambda_{jk}}\beta_{jk}\right)\phi_{\lambda_{ki}}\right] \\
& + [\lambda_{ij}\,(\xi_i + \eta_{ij}) + \beta_{ij}\,\beta_{ji}]\,\phi_{\beta_{ij}} + [\lambda_{ij}\,(\xi_j + \eta_{ij}) + \beta_{ij}\,\beta_{ji}]\,\phi_{\beta_{ji}} \\
& + \sum_1^n{}^k \Bigg\{\left(2\beta_{ij}\,\beta_{jk} - \frac{\rho_{ijk}}{\lambda_{ki}}\beta_{ik}\,\beta_{jk} + \lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji}\right)\phi_{\beta_{jk}} \\
& + \frac{\rho_{ijk}}{\lambda_{kj}}\left(\beta_{jk}\,\beta_{kj} - \lambda_{jk}\,\eta_{jk} - \beta_{ki}\,\beta_{kj}\right)\phi_{\beta_{ki}} + \Bigg(2\beta_{ji}\,\beta_{ik} - \frac{\rho_{ijk}}{\lambda_{jk}}\beta_{ik}\,\beta_{jk} \\
& + \lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji}\Bigg)\phi_{\beta_{ik}} + \frac{\rho_{ijk}}{\lambda_{ki}}\left(\beta_{ki}\,\beta_{ik} - \lambda_{ki}\,\eta_{ki} - \beta_{jk}\,\beta_{ik}\right)\phi_{\beta_{kj}}\Bigg\} = 0, \\
& \qquad (i, j, k = 1, 2, \ldots, n;\; k \neq i,\; k \neq j); \qquad (44)
\end{aligned}
$$

where the upper index λ indicates the result of replacing (40) by (39), and where we have put

$$
\rho_{ijk} = \rho_{jki} = \rho_{kij} = \frac{v_i\,v_j\,v_k}{r_{ij}\,r_{jk}\,r_{ki}} = \sqrt{\rho_{ij}\,\rho_{jk}\,\rho_{ki}\,\rho_{ik}\,\rho_{kj}\,\rho_{ji}} = \sqrt{\lambda_{ij}\,\lambda_{jk}\,\lambda_{ki}}, \qquad (i, j, k = 1, 2, \ldots, n). \qquad (45)
$$

The complete system (41), (42), (43), (44) consists of $n\,(2n - 1)$ equations in $n\,(2n - 1)$ variables

$$
\left.\begin{matrix}
\xi_1, \xi_2, \ldots, \xi_n; & \eta_{12}, \ldots, \eta_{jk}, \ldots, \eta_{n-1\,n}; & \\
\lambda_{12}, \ldots, \lambda_{n-1\,n}; & \beta_{12}, \ldots, \beta_{n-1\,n}; & \beta_{21}, \ldots, \beta_{n-1\,n};
\end{matrix}\right\} \quad (46)
$$

in order that the system have a solution it is necessary and sufficient that the determinant of the coefficients of the partial derivatives should vanish.

Let the equations be so written that the partial derivatives follow the order (46), and let the coefficients be

$$
\left.\begin{matrix}
\mathrm{X}_1^{(i)}, \mathrm{X}_2^{(i)}, \ldots, \mathrm{X}_n^{(i)}; & \mathrm{H}_{12}^{(i)}, \ldots, \mathrm{H}_{n-1\,n}^{(i)}; & \\
\Lambda_{12}^{(i)}, \ldots, \Lambda_{n-1\,n}^{(i)}; & \mathrm{B}_{12}^{(i)}, \ldots, \mathrm{B}_{n-1\,n}^{(i)}; & \mathrm{B}_{21}^{(i)}, \ldots, \mathrm{B}_{n\,n-1}^{(i)},
\end{matrix}\right\} \quad (47)
$$
$$
[i = 1, 2, \ldots, n\,(2n - 1)].
$$

Then it appears at once from the equations that

$$
\sum_1^n{}^j \mathrm{X}_j^{(i)} - 2\sum_1^n{}^j \sum{}^k \mathrm{H}_{jk}^{(i)} = 0, \qquad [i = 1, 2, \ldots, n\,(2n - 1)], \qquad (48)
$$

for all values of i; hence the determinant vanishes.

The form of the equations (41) suggests the possibility of solutions in which the variables

$$\lambda_{ij},\ \beta_{ij},\ \beta_{ji}, \qquad (i, j = 1, 2, \ldots, n), \tag{49}$$

do not occur. To this end it is necessary and sufficient that all determinants of order $\frac{1}{2}n(n+1)$ in the matrix

$$\| \mathrm{X}_1^{(i)}, \mathrm{X}_2^{(i)}, \ldots, \mathrm{X}_n^{(i)}, \mathrm{H}_{12}^{(i)}, \ldots, \mathrm{H}_{jk}^{(i)}, \ldots, \mathrm{H}_{n-1\,n}^{(i)} \|, \ [i = 1, 2, \ldots, n(2n-1)], \tag{50}$$

should vanish; but all these determinants do vanish in consequence of (48). If the system corresponding to any of these determinants be taken, the solution is immediately found in the form

$$\sum_1^n{}^i\, \xi_i - 2\sum_1^n{}^i\sum{}^j\, \eta_{ij}; \tag{51}$$

this integral involves all of the original variables

$$u_i,\ v_i,\ w_i,\ q_{ij},\ r_{ij},\ s_{ij},\ s_{ji}, \quad (i, j = 1, 2, \ldots, n); \tag{52}$$

and hence we have an additional reason for the vanishing of the determinant of the complete system (17).

The further reduction of the complete system of equations which we have been studying is attended by serious complications. It is possible however to examine the most symmetrical case, namely that for which n is equal to three, more closely with comparative ease, and to show by a method which extends itself to the case of n arbitrary, that no other solutions exist than those already found.

When n equals three the reduced system of the $n(2n-1)$th order consists of fifteen equations which can be arranged in five sets of three equations each, the indices being permuted cyclically in each set; the equations for this case are as follows:

$$(v_1 b_1^1 + w_1 c_1^1)^{234} \equiv A_{123} = 0,\ (v_2 b_2^{12} + w_2 c_2^{12})^{34} \equiv A_{231} = 0,\ (v_3 b_3^{123} + w_3 c_3^{123})^4 \equiv A_{312} = 0; \tag{53}$$

$$f_{12}^{1234} \equiv B_{123} = 0, \qquad f_{23}^{1234} \equiv B_{231} = 0, \qquad f_{31}^{1234} \equiv B_{312} = 0; \tag{54}$$

$$(w_1 f_{12}^1 - v_1 e_{21}^1)^{234} \equiv C_{123} = 0,\ (w_2 f_{23}^{12} - v_2 e_{32}^{12})^{34} \equiv C_{231} = 0,\ (w_3 f_{31}^{123} - v_3 e_{13}^{123})^4 \equiv C_{312} = 0; \tag{55}$$

$$(w_1 f_{31}^1 - v_1 e_{31}^1)^{234} \equiv D_{123} = 0,\ (w_2 f_{12}^{12} - v_2 e_{12}^{12})^{34} \equiv D_{231} = 0,\ (w_3 f_{23}^{123} - v_3 e_{23}^{123})^4 \equiv D_{312} = 0; \tag{56}$$

$$\left.\begin{aligned} \{w_2(w_1 f_{12}^1 - v_1 e_{21}^1)^2 - v_2(v_1 c_1^1 + w_1 e_{12}^1)^2\}^{34} &\equiv E_{123} = 0, \\ \{w_3(w_2 f_{23}^{12} - v_2 e_{32}^{12})^3 - v_3(v_2 c_2^{12} + w_2 e_{23}^{12})^3\}^4 &\equiv E_{231} = 0, \\ \{w_3(w_1 f_{31}^1 - v_1 e_{31}^1)^{23} - v_3(v_1 c_3^1 + w_1 e_{13}^1)^{23}\}^4 &\equiv E_{312} = 0; \end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned}
A_{ijk} \equiv{}& 2\beta_{ij}\lambda_{ij}\phi_{\lambda_{ij}} + 2\beta_{ik}\lambda_{ki}\phi_{\lambda_{ki}} + \lambda_{ki}\eta_{ki}\phi_{\beta_{ki}} + (\xi_i + \beta_{ij}^2)\phi_{\beta_{ij}} + \lambda_{ij}\eta_{ij}\phi_{\beta_{ji}} \\
& + (\xi_i + \beta_{ik}^2)\phi_{\beta_{ik}}; \\
B_{ijk} \equiv{}& 2\beta_{ij}\phi_{\xi_i} + 2\beta_{ji}\phi_{\xi_j} + (\beta_{ij} + \beta_{ji})\phi_{\eta_{ij}} + \frac{\lambda_{ij}}{\rho}(\beta_{kj} - \beta_{ki})(\phi_{\eta_{jk}} - \phi_{\eta_{ki}}) \\
& + (\lambda_{ij} - 1)(\phi_{\beta_{ij}} + \phi_{\beta_{ji}}) + \left(\frac{\rho}{\lambda_{ki}} - 1\right)\phi_{\beta_{jk}} + \left(\frac{\rho}{\lambda_{jk}} - 1\right)\phi_{\beta_{ik}}; \\
C_{ijk} \equiv{}& 2(\lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji})\phi_{\xi_i} + 2\xi_j\phi_{\xi_j} + (\xi_j + \lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji})\phi_{\eta_{ij}} \\
& + \frac{\lambda_{ij}}{\rho}(\lambda_{jk}\eta_{jk} - \beta_{jk}\beta_{kj} + \beta_{jk}\beta_{ki})(\phi_{\eta_{jk}} - \phi_{\eta_{ki}}) + 2\lambda_{ij}(1 - \lambda_{ij})\phi_{\lambda_{ij}} \\
& + 2\lambda_{jk}\left(1 - \frac{\rho}{\lambda_{ki}}\right)\phi_{\lambda_{jk}} + (\beta_{ji} - \lambda_{ij}\beta_{ij})\phi_{\beta_{ij}} + \beta_{ji}(1 - \lambda_{ij})\phi_{\beta_{ji}} \\
& + \left(2\beta_{jk} - \beta_{ji} - \frac{\rho}{\lambda_{ki}}\beta_{jk}\right)\phi_{\beta_{jk}} + \frac{\rho}{\lambda_{ki}}(\beta_{ki} - \beta_{kj})\phi_{\beta_{kj}} + \left(\beta_{ji} - \frac{\rho}{\lambda_{jk}}\beta_{jk}\right)\phi_{\beta_{ik}}; \\
D_{ijk} \equiv{}& C_{jik}; \\
E_{ijk} \equiv{}& 2\beta_{ji}\xi_i\phi_{\xi_i} + 2\beta_{ij}\xi_j\phi_{\xi_j} + (\beta_{ji}\xi_i + \beta_{ij}\xi_j)\phi_{\eta_{ij}} \\
& + \frac{\lambda_{ij}}{\rho}\{\beta_{ik}(\lambda_{jk}\eta_{jk} - \beta_{jk}\beta_{kj}) - \beta_{jk}(\lambda_{ki}\eta_{ki} - \beta_{ki}\beta_{ik})\}(\phi_{\eta_{jk}} - \phi_{\eta_{ki}}) \\
& + 2\lambda_{ij}(\beta_{ij} + \beta_{ji})\phi_{\lambda_{ij}} + 2\lambda_{jk}\left(\beta_{ij} - \frac{\rho}{\lambda_{ki}}\beta_{ik}\right)\phi_{\lambda_{jk}} - 2\lambda_{ki}\left(\beta_{ji} - \frac{\rho}{\lambda_{jk}}\beta_{jk}\right)\phi_{\lambda_{ki}} \\
& + [\lambda_{ij}(\xi_i + \eta_{ij}) + \beta_{ij}\beta_{ji}]\phi_{\beta_{ij}} + \left(\lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji} + 2\beta_{ij}\beta_{jk} - \frac{\rho}{\lambda_{ki}}\beta_{ik}\beta_{jk}\right)\phi_{\beta_{jk}} \\
& + \frac{\rho}{\lambda_{jk}}(\beta_{kj}\beta_{jk} - \lambda_{jk}\eta_{jk} - \beta_{jk}\beta_{ki})\phi_{\beta_{ki}} + \Big(\lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji} + 2\beta_{ji}\beta_{ik} \\
& - \frac{\rho}{\lambda_{jk}}\beta_{ik}\beta_{jk}\Big)\phi_{\beta_{ik}} + \frac{\rho}{\lambda_{ki}}(\beta_{ki}\beta_{ik} - \lambda_{ki}\eta_{ki} - \beta_{ik}\beta_{kj})\phi_{\beta_{kj}} \\
& + [\lambda_{ij}(\xi_j + \eta_{ij}) + \beta_{ij}\beta_{ji}]\phi_{\beta_{ji}};
\end{aligned}\right\} \qquad (58)$$

where

$$\rho = \sqrt{\lambda_{ij}\lambda_{jk}\lambda_{ki}}. \qquad (59)$$

That the above fifteen equations in fifteen variables possess at least one solution appears from the fact that we have

$$P_{ijk} + P_{jki} + P_{kij} = 0, \qquad (60)$$

where

$$P_{ijk} \equiv A_{ijk} + \frac{1}{\lambda_{ij}}\{(\beta_{ij}\beta_{ji} - \lambda_{ij}\eta_{ij})B_{ijk} + \beta_{ij}C_{ijk} + \beta_{ji}D_{ijk} - E_{ijk}\}. \qquad (61)$$

That they form a complete system is verified by reference to the table below, from which have been omitted all vanishing commutators and all cyclical changes of those given:

$$\left.\begin{aligned}
&(A_{ijk},\ B_{ijk}) \equiv \beta_{ij}\, B_{ijk} + D_{ijk};\quad (A_{ijk},\ B_{kij}) \equiv \beta_{ik}\, B_{kij} + C_{kij};\\
&(A_{ijk},\ C_{ijk}) \equiv \beta_{ij}\, C_{ijk} - E_{ijk};\quad (A_{ijk},\ C_{kij}) \equiv \beta_{ik}\, C_{kij} - \xi_i\, B_{kij};\\
&(A_{ijk},\ D_{ijk}) \equiv \beta_{ij}\, C_{ijk} - E_{ijk};\quad (A_{ijk},\ D_{kij}) \equiv \beta_{ik}\, D_{kij} - E_{kij};\\
&(A_{ijk},\ E_{ijk}) \equiv \xi_i\, C_{ijk} + \beta_{ij}\, E_{ijk} - 4\,\beta_{ji}\, A_{ijk};\quad (A_{ijk},\ E_{kij}) \equiv \xi_i\, D_{kij} + \beta_{ik}\, E_{kij}\\
&\qquad\qquad - 4\,\beta_{ki}\, A_{ijk};\\
&(B_{ijk},\ C_{ijk}) \equiv (1+\lambda_{ij})\, B_{ijk} \equiv (B_{ijk},\ D_{ijk});\ (B_{ijk},\ C_{kij}) \equiv B_{kij} + \frac{\rho}{\lambda_{jk}}(B_{ijk} - B_{jki});\\
&(B_{ijk},\ D_{jki}) \equiv B_{jki} + \frac{\rho}{\lambda_{ki}}(B_{ijk} - B_{kij});\quad (B_{ijk},\ E_{ijk}) \equiv -C_{ijk} - D_{ijk};\\
&(B_{ijk},\ E_{jki}) \equiv \frac{\rho}{\lambda_{ki}}\,\beta_{ki}\, B_{ijk} - C_{jki} + \frac{\rho}{\lambda_{ki}}\, D_{kij};\quad (B_{ijk},\ E_{kij}) \equiv \frac{\rho}{\lambda_{jk}}\,\beta_{kj}\, B_{ijk} + C_{jki}\\
&\qquad\qquad - \frac{\rho}{\lambda_{jk}}\, D_{kij};\\
&(C_{ijk},\ C_{jki}) \equiv \beta_{kj}\, B_{ijk} + \frac{\rho}{\lambda_{ki}}(D_{kij} - C_{jki});\quad (C_{ijk},\ D_{ijk}) \equiv (\beta_{ij} - \beta_{ji})\, B_{ijk}\\
&\qquad\qquad + \lambda_{ij}\,(C_{ijk} - D_{ijk});\\
&(C_{ijk},\ D_{jki}) \equiv \beta_{ji}\, B_{jki} - \beta_{jk}\, B_{ijk} + \left(\frac{\rho}{\lambda_{ki}} - 2\right)(C_{ijk} - D_{jki});\\
&(C_{ijk},\ E_{ijk}) \equiv \lambda_{ij}\,(A_{ijk} + 3\,A_{jki}) + (\beta_{ij} - \beta_{ji})\, C_{ijk} + (1 - \lambda_{ij})\, E_{ijk};\\
&(C_{ijk},\ E_{jki}) \equiv (\lambda_{jk}\,\eta_{jk} - \beta_{jk}\,\beta_{kj})\, B_{ijk} + \left(\frac{\rho}{\lambda_{ki}}\beta_{ki} - 2\,\beta_{kj}\right) C_{ijk} - \beta_{ji}\, C_{jki}\\
&\qquad\qquad + \left(2 - \frac{\rho}{\lambda_{ki}}\right) E_{jki};\\
&(C_{ijk},\ E_{kij}) \equiv \frac{\rho}{\lambda_{jk}}\beta_{kj}\, C_{ijk} + \beta_{ji}\, D_{kij} - \frac{\rho}{\lambda_{jk}}\, E_{jki};\\
&(D_{ijk},\ D_{jki}) \equiv -\beta_{ij}\, B_{jki} - \frac{\rho}{\lambda_{ki}}(C_{kij} - D_{ijk});\\
&(D_{ijk},\ D_{kij}) \equiv \beta_{ki}\, B_{ijk} + \frac{\rho}{\lambda_{jk}}(C_{jki} - D_{kij});\\
&(D_{ijk},\ E_{ijk}) \equiv \lambda_{ij}\,(3\,A_{ijk} + A_{jki}) + (\beta_{ij} - \beta_{ji})\, D_{ijk} + (1 - \lambda_{ij})\, E_{ijk};\\
&(D_{ijk},\ E_{jki}) \equiv \beta_{ij}\, C_{jki} + \frac{\rho}{\lambda_{ki}}\beta_{ki}\, D_{ijk} - \frac{\rho}{\lambda_{ki}}\, E_{kij};\\
&(D_{ijk},\ E_{kij}) \equiv (\lambda_{ki}\,\eta_{ki} - \beta_{ki}\,\beta_{ik})\, B_{ijk} + \left(\frac{\rho}{\lambda_{jk}}\beta_{kj} - 2\,\beta_{ki}\right) D_{ijk} - \beta_{ij}\, D_{kij}\\
&\qquad\qquad + \left(2 - \frac{\rho}{\lambda_{jk}}\right) E_{kij};\\
&(E_{ijk},\ E_{jki}) \equiv (\lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji})\, C_{jki} - (\lambda_{jk}\,\eta_{jk} - \beta_{jk}\,\beta_{kj})\, D_{ijk}\\
&\qquad\qquad + \left(\frac{\rho}{\lambda_{ki}}\,\beta_{ki} - 2\,\beta_{kj}\right) E_{ijk} + \left(2\,\beta_{ij} - \frac{\rho}{\lambda_{ki}}\,\beta_{ik}\right) E_{jki};\\
&(E_{ijk},\ E_{kij}) \equiv (\beta_{ki}\,\beta_{ik} - \lambda_{ki}\,\eta_{ki})\, C_{ijk} + (\lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji})\, D_{kij}\\
&\qquad\qquad + \left(\frac{\rho}{\lambda_{jk}}\,\beta_{kj} - 2\,\beta_{ki}\right) E_{ijk} + \left(2\,\beta_{ji} - \frac{\rho}{\lambda_{jk}}\,\beta_{jk}\right) E_{kij}.
\end{aligned}\right\}\ (62)$$

The system is known to possess two solutions, namely (4) and (51) from the preceding discussion. The latter of these is

$$\xi_i + \xi_j + \xi_k - 2(\eta_{ij} + \eta_{jk} + \eta_{ki}); \tag{63}$$

the former may be written

$$\begin{vmatrix} u_i & w_i & q_{ji} & s_{ij} & q_{ki} & s_{ik} \\ w_i & v_i & s_{ji} & r_{ij} & s_{ki} & r_{ik} \\ q_{ij} & s_{ji} & u_j & w_j & q_{kj} & s_{jk} \\ s_{ij} & r_{ji} & w_j & v_j & s_{kj} & r_{jk} \\ q_{ik} & s_{ki} & q_{jk} & s_{kj} & u_k & w_k \\ s_{ik} & r_{ki} & s_{jk} & r_{kj} & w_k & v_k \end{vmatrix} \tag{64}$$

Designate by 123456 the columns of the matrix formed by the first two rows of this determinant (64); those of the matrix formed by the third and fourth rows by $1'2'3'4'5'6'$; and finally the columns of the matrix formed by the last two rows by $1_1 2_1 3_1 4_1 5_1 6_1$; call these matrices A, B, C respectively.

The expressions of the minors of any one of them, say A, in the new variables, can be determined by means of the relations (37) and (38); the expressions of the remaining ones may then be written down by cyclical permutation guided by the following substitution scheme:

$$A(123456) \quad B(3'4'5'6'1'2') \quad C(5_1 6_1 1_1 2_1 3_1 4_1); \tag{65}$$

and the expansion of the determinant by the method of Laplace be obtained by substitution in the symbolical scheme below:

$$\left.\begin{array}{l}
\ 12\,\{3456 + 4536 + 5634 - 3546 - 4635 + 3645\} \\
+ 23\,\{1456 + 4516 + 5614 - 1546 - 4615 + 1645\} \\
+ 34\,\{1256 + 2516 + 5612 - 1526 - 2615 + 1625\} \\
+ 45\,\{1236 + 2316 + 3612 - 1326 - 2613 + 1623\} \\
+ 56\,\{1234 + 2314 + 3412 - 1324 - 2413 + 1423\} \\
- 13\,\{2456 + 4526 + 5624 - 2546 - 4625 + 2645\} \\
- 24\,\{1356 + 3516 + 5613 - 1536 - 3615 + 1635\} \\
- 35\,\{1246 + 2416 + 4612 - 1426 - 2614 + 1624\} \\
- 46\,\{1235 + 2315 + 3512 - 1325 - 2513 + 1523\} \\
+ 14\,\{2356 + 3526 + 5623 - 2536 - 3625 + 2635\} \\
+ 25\,\{1346 + 3416 + 4613 - 1436 - 3614 + 1634\} \\
+ 36\,\{1245 + 2415 + 4512 - 1425 - 2514 + 1524\} \\
- 15\,\{2346 + 3426 + 4623 - 2436 - 3624 + 2634\} \\
- 26\,\{1345 + 3415 + 4513 - 1435 - 3514 + 1534\} \\
+ 16\,\{2345 + 3425 + 4523 - 2435 - 3524 + 2534\},
\end{array}\right\} \tag{66}$$

in which the numbers without the parentheses belong to A, and of those within the parentheses the first pair of each set belongs to B and the second pair to C; the resulting form is long and complicated, the elegance of the form (64) disappearing in the transformation, and as it is unnecessary to our purposes it need not be reproduced here.

It is now proposed to show by the aid of the determinant of the partial derivatives of the system (58) that the system composed of (53), (54), (55), (56), (57) does not possess more than two solutions.

Call the determinant D and write it down so that its fifteen columns proceed in the order of the partial derivatives with regard to

$$\xi_i,\ \xi_j,\ \xi_k,\ \eta_{ij},\ \eta_{jk},\ \eta_{ki},\ \lambda_{ij},\ \lambda_{jk},\ \lambda_{ki},\ \beta_{ij},\ \beta_{jk},\ \beta_{ki},\ \beta_{ji},\ \beta_{kj},\ \beta_{ik}, \tag{67}$$

respectively, and the rows in the order of the respective equations

$$A_{ijk},\ A_{jki},\ A_{kij},\ B_{ijk},\ B_{jki},\ B_{kij},\ C_{ijk},\ C_{jki},\ C_{kij},\ D_{ijk},\ D_{jki},\ D_{kij},\ E_{ijk},\ E_{jki},\ E_{kij}. \tag{68}$$

Consider now the subdeterminant of the thirteenth order of D formed by cutting out the fifth and sixth columns and dropping the fourteenth and fifteenth rows; designate this subdeterminant by E, and its columns by C_i and rows by R_j, and by $E_{i,j}$ the element common to C_i and R_j.

It is not difficult to isolate a unique non-vanishing term in E and thus prove that E itself does not vanish.

To this end transform the determinant E by the following operations: 1) Replace ξ_j by zero; 2) replace all the β's by zeros except β_{ji}; 3) from C_{10} subtract C_{12}; 4) from R_{12} subtract R_8; 5) from C_{11} subtract $\beta_{ji} C_5$; 6) from C_4 take $\frac{\lambda_{jk}}{\rho}\,C_1$; 7) from R_{13} take $\frac{\rho}{\lambda_{jk}}\,R_1$. In the resulting determinant ξ_k occurs only at $E_{3,8}$ and $E_{12,3}$; accordingly we have the coefficient of ξ_k^2 by suppressing R_3, R_8, C_3 and C_{12}. In the thus depleted R_1 of E the term $\lambda_{ki}\eta_{ki}$ occurs only at $E_{10,1}$; in the depleted C_1, $\lambda_{ki}\eta_{ki}$ occurs only at $E_{1,12}$; in the depleted R_2 the term $\lambda_{ij}\eta_{ij}$ occurs only at $E_{8,2}$; and in the depleted C_2 the term $\lambda_{ij}\eta_{ij}$ appears only at $E_{2,10}$; accordingly, by suppressing further from E, R_1, R_2, R_{10}, R_{12}, C_1, C_2, C_8, C_{10}, we have the coefficient of $(\xi_k\lambda_{ki}\eta_{ki}\lambda_{ij}\eta_{ij})^2$; in this last-named coefficient there occurs a unique element, $\beta_{ji}\xi_i$, at $E_{4,13}$ of the original determinant E. By these successive steps the coefficient of $\beta_{ji}\xi_i(\xi_k\lambda_{ki}\eta_{ki}\lambda_{ij}\eta_{ij})^2$ is made to appear in the following form:

$$\begin{array}{cccccccc}
C_5 & C_6 & C_7 & C_9 & C_{11} & C_{13} & \\
0 & 0 & 0 & \frac{\rho}{\lambda_{ki}}-1 & \lambda_{ij}-1 & \frac{\rho}{\lambda_{jk}}-1 & R_4 \\
0 & 0 & 0 & \lambda_{jk}-1 & \frac{\rho}{\lambda_{ki}}-1 & 0 & R_5 \\
0 & 0 & 0 & 0 & 0 & \lambda_{ki}-1 & R_6 \\
1-\lambda_{ij} & 1-\frac{\rho}{\lambda_{ki}} & 0 & -\beta_{ji} & 0 & \beta_{ji} & R_7 \\
1-\frac{\rho}{\lambda_{jk}} & 0 & 1-\lambda_{ki} & 0 & -\beta_{ji} & 0 & R_9 \\
1-\frac{\rho}{\lambda_{ki}} & 1-\lambda_{jk} & 0 & 0 & \beta_{ji} & 0 & R_{11}
\end{array} \tag{69}$$

Hence we see that the term

$$\beta_{ji}\xi_i\left\{\xi_k\lambda_{ki}\eta_{ki}\lambda_{ij}\eta_{ij}(1-\lambda_{ki})\begin{vmatrix}\lambda_{ij}-1 & \frac{\rho}{\lambda_{ki}}-1 \\ \frac{\rho}{\lambda_{ki}}-1. & \lambda_{jk}-1\end{vmatrix}\right\}^2 \tag{70}$$

occurs but once in the determinant D.

We may conclude then that the system of equations possesses no more than two independent solutions.

It is not difficult to write out the extension of the argument of the last paragraph to the case of n arbitrary. To this end consider the determinant of the twenty-eighth order arising in the group associated with the problem of five bodies, and write its columns in the order of the partial derivatives with regard to the variables

$$\left.\begin{array}{l}\xi_i,\ \xi_j,\ \xi_k,\ \xi_l,\ \eta_{ij},\ \eta_{jk},\ \eta_{ki},\ \eta_{jl},\ \eta_{li},\ \eta_{kl},\ \lambda_{ij},\ \lambda_{jk},\ \lambda_{ki},\ \lambda_{jl},\ \lambda_{li},\ \lambda_{kl}, \\ \beta_{ij},\ \beta_{jk},\ \beta_{ki},\ \beta_{jl},\ \beta_{li},\ \beta_{kl},\ \beta_{ji},\ \beta_{kj},\ \beta_{ik},\ \beta_{lj},\ \beta_{il},\ \beta_{lk},\end{array}\right\} \tag{71}$$

respectively, and its rows in the order of the equations (41), (42), (43), (44), namely

$$\left.\begin{array}{l}34i,\ 34j,\ 34k,\ 34l,\ 33ij,\ 33jk,\ 33ki,\ 33jl,\ 33li,\ 33kl,\ 35ij,\ 35jk,\ 35ki, \\ 35jl,\ 35li,\ 35kl,\ 35ji,\ 35kj,\ 35ik,\ 35lj,\ 35il,\ 35lk,\ 36ij,\ 36jk,\ 36ki, \\ 36jl,\ 36li,\ 36kl,\end{array}\right\} \tag{72}$$

respectively.

Consider now the minor of this determinant formed by cutting out $C\eta_{jk}$, $C\eta_{ki}$, $R36jk$, $R36ki$. Retaining the designations of the rows and columns as in the original determinant let us transform this minor by the following steps: 1) From $C\beta_{ki}$ and $C\beta_{kl}$ take $C\beta_{kj}$; 2) from $R36kl$ take $\beta_{lk}\,R35kl$; 3) from $R35ki$ subtract $R35kj$; 4) from $R35kl$ subtract $R35kj$; 5) from $R35kl$ take $R35kj$. In the first four columns of the depleted determinant ξ_k occurs only at $(R35kj,\ C\xi_k)$; in the first column p_{ki} occurs only at $R35ki$; in the second column p_{ij} occurs only at $R35ij$; in the fourth column p_{il} occurs only at $R35il$. From $C\xi_l$ take $C\eta_{lk}$; from $C\xi_j$ take $C\eta_{ij}$. We have the coefficient of $\xi_k\,p_{ij}\,p_{ki}\,p_{il}$ by cutting out the first four columns and the rows $R35kj$, ki, ij, il. In the first row p_{ki} occurs only at $C\beta_{ki}$; in the second row p_{ij} occurs only at $C\beta_{ij}$; in the fourth row p_{il} occurs only at $C\beta_{il}$; in the first four rows ξ_k occurs only at $(R34k,\ C\beta_{kj})$. We have the coefficient of $\xi_k^2\ p_{ij}^2\ p_{ki}^2\ p_{il}^2$ by cutting off further the first four rows, and the columns $C\beta kj$, ki, ij, il. In the last-named coefficient ξ_k occurs only at $(R36kl,\ C\beta_{kl})$ with multiplier λ_{kl}, and at $(R35kl,\ C\eta_{kl})$; at $(R36li,\ C\beta_{jl})$ we have the term p_{ij} multiplied by $\left(-\frac{\rho_{lij}}{\lambda_{jl}}\right)$, and p_{ij} occurs at no other point in that row or column; at $(R35jl,\ C\eta_{li})$ we have the term p_{ij} multiplied by $\left(-\frac{\lambda_{jl}}{\rho_{ijl}}\right)$, and p_{ij} occurs at no other point in that row or column; cut out then additionally $R36kl$, $R35kl$, $R36li$, $R35jl$, $C\beta_{kl}$, $C\eta_{kl}$, $C\beta_{jl}$, $C\eta_{li}$, and we have the coefficient of $\lambda_{kl}\,\xi_k^4\,p_{ij}^4\,p_{ki}^2\,p_{il}^2$; in this coefficient $\xi_i\,\beta_{ji}$ is a unique term at $(R36ij,\ C\eta_{ij})$, and $\xi_j\,\beta_{lj}$ is a unique term and at $(R36jl,\ C\eta_{jl})$; accordingly, we have a unique term $\lambda_{kl}\,\beta_{ji}\,\beta_{lj}\,\xi_i\,\xi_j\,\xi_k^4\,p_{ij}^4\,p_{ki}^2\,p_{il}^2$ whose coefficient is the determinant formed of the elements common to

$$\left.\begin{array}{ll} R35jk,\ li,\ ji,\ ik,\ lj,\ lk, & 33ij,\ jk,\ ki,\ jl,\ li,\ kl, \\ C\beta jk,\ li,\ ji,\ ik,\ lj,\ lk, & \lambda ij,\ jk,\ ki,\ jl,\ li,\ kl. \end{array}\right\} \tag{73}$$

Examining this twelfth order determinant we remark: 1) The elements of the minor

$$\left.\begin{array}{l} R33 \\ C\lambda \end{array}\right\} ij,\ jk,\ ki,\ jl,\ li,\ kl \tag{74}$$

are all zero; 2) as regards the elements of the minor

$$\left.\begin{array}{l} R35 \\ C\beta \end{array}\right\} jk,\ li,\ ji,\ ik,\ lj,\ lk, \tag{75}$$

each involves the β's and λ's only; 3) the minors

$$\begin{matrix} R33 \; ij, jk, ki, jl, li, kl; \\ C\beta \; jk, li, ji, ik, lj, lk \end{matrix} \tag{76}$$

and

$$\begin{matrix} R35 \; jk, li, ji, ik, lj, lk; \\ C\lambda \; ij, jk, ki, jl, li, kl \end{matrix} \tag{77}$$

involve the λ's only, and are equal, the rows of one being the columns of the other, and reciprocally.

Accordingly we seek to determine whether the determinant

$$\begin{vmatrix} 1-\frac{\rho_{ijk}}{\lambda_{ki}} & 0 & 1-\lambda_{ij} & 1-\frac{\rho_{ijk}}{\lambda_{jk}} & 0 & 0 \\ 1-\lambda_{jk} & 0 & 1-\frac{\rho_{ijk}}{\lambda_{ki}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda_{ki} & 0 & 0 \\ 1-\frac{\rho_{jkl}}{\lambda_{kl}} & 1-\frac{\rho_{jli}}{\lambda_{ij}} & 1-\frac{\rho_{jli}}{\lambda_{li}} & 0 & 1-\lambda_{jl} & 1-\frac{\rho_{jkl}}{\lambda_{jk}} \\ 0 & 1-\lambda_{li} & 0 & 1-\frac{\rho_{lik}}{\lambda_{kl}} & 1-\frac{\rho_{lij}}{\lambda_{ij}} & 1-\frac{\rho_{lik}}{\lambda_{ki}} \\ 0 & 1-\frac{\rho_{kli}}{\lambda_{ki}} & 0 & 0 & 1-\frac{\rho_{jkl}}{\lambda_{jk}} & 1-\lambda_{kl} \end{vmatrix} \tag{78}$$

vanishes or not.

This last determinant may be written as follows:

$$(1-\lambda_{ki})\begin{vmatrix} 1-\frac{\rho_{ijk}}{\lambda_{ki}} & 1-\lambda_{ij} \\ 1-\lambda_{jk} & 1-\frac{\rho_{ijk}}{\lambda_{ki}} \end{vmatrix} \cdot \begin{vmatrix} 1-\frac{\rho_{jli}}{\lambda_{ij}} & 1-\lambda_{jl} & 1-\frac{\rho_{jkl}}{\lambda_{jk}} \\ 1-\lambda_{li} & 1-\frac{\rho_{lij}}{\lambda_{ij}} & 1-\frac{\rho_{lik}}{\lambda_{ki}} \\ 1-\frac{\rho_{kli}}{\lambda_{ki}} & 1-\frac{\rho_{jkl}}{\lambda_{jk}} & 1-\lambda_{kl} \end{vmatrix}, \tag{79}$$

in which form its non-vanishing is obvious.

For the general case it is in similar manner made to appear that there is a non-vanishing minor of order $2n^2-n-2$ containing the un m made up of the product of the following terms:

$$(\lambda_{s_1s_4}\lambda_{s_1s_5}\ldots\lambda_{s_1s_n})(\xi_{s_2}\beta_{s_3s_2}\xi_{s_3}\beta_{s_4s_3}\ldots\xi_{s_{n-1}}\beta_{s_n s_{n-1}})(p^2_{s_1s_2}\xi^{2(n-2)}_{s_1}p^{2(n-2)}_{s_2s_3}p^2_{s_2} \quad p^2_{s_2s_n}) \quad (80)$$

$$(1-\lambda_{s_1s_2})^2\cdot\begin{vmatrix}1-\lambda_{s_1s_3} & 1-\frac{\rho_{s_1s_2s_3}}{\lambda_{s_1s_2}}\\ 1-\frac{\rho_{s_1s_2s_3}}{\lambda_{s_1s_2}} & 1-\lambda_{s_2s_3}\end{vmatrix}\cdot\prod_{i=4}^{i=n}\begin{vmatrix}1-\lambda_{s_1s_i} & 1-\frac{\rho_{s_2s_1s_i}}{\lambda_{s_2s_1}} & \ldots & 1-\frac{\rho_{s_{i-1}s_1s_i}}{\lambda_{s_{i-1}s_1}}\\ 1-\frac{\rho_{s_1s_2s_i}}{\lambda_{s_1s_2}} & 1-\lambda_{s_2s_i} & \ldots & 1-\frac{\rho_{s_{i-1}s_2s_i}}{\lambda_{s_{i-1}s_2}}\\ 1-\frac{\rho_{s_1s_3s_i}}{\lambda_{s_1s_3}} & 1-\frac{\rho_{s_2s_3s_i}}{\lambda_{s_2s_3}} & \ldots & 1-\frac{\rho_{s_{i-1}s_3s_i}}{\lambda_{s_{i-1}s_3}}\\ \ldots & \ldots & \ldots & \ldots\\ 1-\frac{\rho_{s_1s_{i-1}s_i}}{\lambda_{s_1s_{i-1}}} & 1-\frac{\rho_{s_2s_{i-1}s_i}}{\lambda_{s_2s_{i-1}}} & \ldots & 1-\lambda_{s_{i-1}s_i}\end{vmatrix} \quad (81)$$

where p_{ij} is written short for $\lambda_{ij}\eta_{ij}-\beta_{ij}\beta_{ji}$.

HOUSTON, TEXAS.

Normal Curves of Genus 6, and their Groups of Birational Transformations.

BY VIRGIL SNYDER.

1. The canonical form to which an algebraic curve of given genus can be reduced is one of the fundamental problems in the theory of birational transformations. The simplest forms of curves of genus 3 and their corresponding groups have been found by Wiman,* who also made a study of space curves of genus 4, and an outline of that of curves of genus 5. The forms and properties of plane curves of genus 4 have been determined by Miss Van Benschoten;† the classification of those of genus 5 is well under way. The present paper has for its purpose the determination of the groups of birational transformations which leave curves of genus 6 invariant and to discuss various properties of such curves. This configuration is interesting from the fact that it is the lowest genus which can not be defined by the complete intersection of quadric spreads in hyperspace, and that only one of the defining spreads can be assumed at will.

The general curve of genus 6 can be reduced to a sextic c_6 with four double points. The only exceptions are the hyperelliptic curve and the non-singular quintic. When the curve is reduced to another of the same order by a non-linear transformation it must contain a linear g_6^2, of which the points of each group are not collinear. Since this is a special series, it can be determined by adjoint cubic curves ϕ_3. But the $\infty^2 \phi_3$ having the four double points and any other three points of c_6 for basis points define not a g_6^2, but g_7^2; hence: *all transformations which transform a non-hyperelliptic curve of genus 6 and order 6 into itself or any other curve of the same order birationally can be expressed by collineations and quadric inversions.* Moreover, every transformation generated by these must, in this case, be either linear or quadratic.

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* *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI (1895).

† A. L. Van Benschoten, *On the Transformations Which Leave the Algebraic Curves of Genus 4 Invariant*, Cornell dissertation, 1908.

The following cases are to be considered:

(a) The normal curve is a c_6 with four double points $(4\,P_2)$ at the vertices of a quadrangle.

(b) The c_6 has three collinear double points, and one other one.

(c) The c_6 has a triple point P_3 and a double point.

(d) The curve has a g_5^2.

(e) The curve is hyperelliptic.

§ 1 (a). *c_6 with Four Double Points, General Case.*

2. This curve has $5\,g_4^1$, formed by the pencils of straight lines through each of the nodes, and the pencil of conics through all of them. When more than five g_4^1 exist, the curve has an infinite number of such series and can not be reduced to a sextic. These series must permute among themselves; hence, *curves of form* (a) *can have no group of order larger than* 120. Let the four double points be (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1). The $\infty^3\,\phi_3$ through these points will be of the form

$$a\,(x^2y-xyz)+b\,(xy^2-xyz)+c\,(x^2z-xyz)+d\,(y^2z-xyz)$$
$$+\,e\,(xz^2-xyz)+f\,(yz^2-xyz)=0.$$

If we now put

$$\rho x_1=xy\,(x-z),\quad \rho x_2=xy\,(y-z),\quad \rho x_3=xz\,(x-y),$$
$$\rho x_4=yz\,(y-x),\quad \rho x_5=xz\,(z-y),\quad \rho x_6=yz\,(z-x),$$

then between the x_i exist the five following identities:

$$\left.\begin{array}{ll} x_2x_5-x_4x_5-x_2x_6=0, & x_3x_6+x_1x_5-x_1x_6=0,\\ x_1x_4+x_2x_3-x_3x_4=0, & x_1x_4-x_1x_6+x_2x_5=0,\\ x_4x_5+x_3x_6-x_3x_4=0. & \end{array}\right\}\quad(1)$$

The equation of any c_6 having the above points for double points can be expressed as a quadratic relation among the x_i. It contains 21 terms, or 20 constants, but five of these can be expressed in terms of the others by means of (1). Thus all the $15=3\,p-3$ constants appear in the one equation

$$\Sigma\, a_{ik}\,x_i\,x_k=0. \qquad (2)$$

Now consider x_i as homogeneous point coordinates in linear space of five dimensions R_5. The systems (1) and (2) define six quadric spreads which have a curve in common. Any ϕ_3 cuts c_6 in 10 points, hence an R_4 defined by $\Sigma\, b_i x_i=0$ will cut the normal curve in 10 points. If the $c_{10}^{(5)}$ be projected from any point upon it and the projecting cone cut by an R_4, the resulting $c_9^{(4)}$ can not have a double point. In other words, $c_{10}^{(5)}$ can have no trisecants. By further projections this curve becomes $c_8^{(3)}$ and $c_7^{(2)}$ respectively. A c_7 with 9 double points can not be reduced to a c_6 by means of ϕ_4 passing through an arbitrary set of basis points, but if one simple basis point be assumed, three others can be found in five different ways, such that $\infty^2\,\phi_4$ can be passed through them and the nine double points.[*]

If these curves be used for the transforming system, the transformed curve will be a c_6 with four double points. Since $c_{10}^{(5)}$ can be projected from certain points upon it into $c_6^{(2)}$, it is possible to find tetrads of points upon it such that through them can be passed $\infty^2\,R_4$. It is necessary and sufficient that the four basis points lie in R_2. If $c_{10}^{(5)}$ be projected from one of them into $c_9^{(4)}$, the other three will go into points lying on a straight line; hence $c_9^{(4)}$ *has at least five trisecants.* If $c_9^{(4)}$ be projected into R_3 from one of the points of intersection with a trisecant, $c_8^{(3)}$ will have a double point. Finally, $c_8^{(3)}$ is projected from the double point into our plane c_6.

3. Let the systems (1) and (2) which define $c_{10}^{(5)}$ be denoted by $F_1, \dots, F_6$. Among the spreads of the linear system $\Sigma\lambda_i F_i = 0$ are ∞^4 which can be expressed in terms of five variables. These particular spreads are the four-dimensional quadric cones, having a point for vertex. The associated values of λ_i are found by equating the discriminant of the system $\Delta(\lambda_1, \dots, \lambda_6) = \Delta(\lambda)$ to zero. If λ_i be regarded as point coordinates in R_5, $\Delta(\lambda) = 0$ is a four-dimensional spread of order 6. The corresponding locus of the vertex of the cones is obtained by equating the determinant of $F_{ik} = \dfrac{\partial F_i}{\partial x_k}$ of order 6 to zero. It is also of order 6. Between this spread M and Δ exists a one-to-one point correspondence. If from a point of M the curve $c_{10}^{(5)}$ be projected into R_4, $c_{10}^{(4)}$ will lie on an $F_2^{(3)}$. But there are values of λ for which F can be expressed in terms of four variables. These spreads are four-dimensional quadric cones having a line for vertex. The values of λ are obtained by equating all the first minors of Δ to zero. The corresponding configuration on M is a ruled hypersurface S. If $c_{10}^{(5)}$ be projected into R_3 from a line of S which is a bisecant of the curve, the $c_8^{(3)}$ is of type (4, 4) on a quadric surface, and has three actual double points.[†]

In no case can $c_9^{(4)}$ have an actual double point, as it would give rise to $c_7^{(3)}$; but this curve has a g_3^1, hence belongs to those of type (c). Through every point of $c_{10}^{(5)}$ can be passed five R_2, each of which cuts the curve in three remaining points. If $c_{10}^{(5)}$ be projected from such an R_2 into R_2, the result is $c_6^{(2)}$ with four double points. Thus the g_6^2 in the $c_6^{(2)}$, and some fixed point on th[illegible]e, can never be a partial series g_7^2 contained in g_7^3, but two such points ca[illegible] that the resulting g_8^2 is a partial series contained in the $g_8^{(3)}$.*

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[*] Clebsch, *Geometrie*, Vol. I, p. 695.

[†] See Riemann, in *Crelle*, Vol. LIV, § 13; Clebsch, *Geometrie*, Vol. I, p. 693, foot-note; Brill, *Math. Ann.*, Vol. I, p. 401, and Vol. II, p. 471.

4. The four basis points project on $c_6^{(2)}$ into the four points in which a conic through the double points cuts the curve. One of them is arbitrary and the others are then fixed. Similarly, the two fixed points which are the images of the double point of $c_8^{(3)}$ are the residual points in which the line joining two nodes cuts the curve again. The adjoint ϕ_3 are made up of the straight line joining the other two nodes, and the ∞^3 conics through the first two; as subgroup we have the degraded conics formed by the line joining the second pair of nodes and an arbitrary line of the plane.

5. The curve $c_{10}^{(5)}$ is a double curve on M, the Jacobian of the system of quadrics. Among the lines S, some are bisecants, some simple secants, but in general they do not intersect $c_{10}^{(5)}$. If the curve be projected from a general line of S, its image in R_3 is a $c_{10}^{(3)}$ of type (5, 5) on a quadric. It has 10 actual double points. On the other hand, if $c_{10}^{(5)}$ be projected into R_3 from any bisecant, the resulting $c_8^{(3)}$ will have no double points. It is the partial intersection of a cubic and a quartic surface, the residual being a rational $c_4^{(3)}$.

Given any point P on $c_{10}^{(5)}$. Associated with it are five sets of three points each, P_1^k, P_2^k, P_3^k $(k = 1, \ldots, 5)$, such that each set and the point P lie in a plane. If these points be called a particular group, we may say: *The* $c_8^{(3)}$ *obtained by projecting* $c_{10}^{(5)}$ *into* R_3 *from a line joining any two points of a particular group will always have at least one double point.*

If $P_1^1 = P_2^1$, then through the line PP_1^1 can be passed two R_2, each cutting $c_{10}^{(5)}$ in two other points. The $c_8^{(3)}$ obtained by projecting from such a line must have at least two double points; but since a g_3^1 on $c_{10}^{(5)}$ is excluded, if $c_8^{(3)}$ has two double points, it has three. Since the points associated with P_1^1 must be P, and the two remaining associates of P, we now have the following theorem:

The necessary and sufficient condition that a line joining two corresponding points of the same particular group on $c_{10}^{(5)}$ *be the vertex of a quadric cone on which* $c_{10}^{(5)}$ *lies is that one point on* $c_{10}^{(5)}$ *is common to two particular groups belonging to the other.*

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* This is a direct application of Noether's theorem of reduction. See Segre, "Introduzione alla Geometria sopra un Ente Algebrico Semplicemente Infinito," *Ann. di Mat.* (2), Vol. XXII (1894).

Thus, there can never be a simple coincidence of associated points on $c_{10}^{(5)}$. In each case the coincidences appear in sets of three.

6. Every non-hyperelliptic curve of genus p greater than 3 has one or more linear g_{p-1}^1. By the Riemann-Roch theorem the residual series is also a g_{p-1}^1. In the canonical curve in R_{p-1} these series must be cut by R_{p-2}, arranged in reciprocal sets. A series of R_{p-3} can be found having $p-1$ points on $c_{2p-2}^{(p-1)}$. Any R_{p-2} through these points will cut the curve in $p-1$ further points, which also lie on a R_{p-3}. The curve lies on a quadric spread in R_{p-1} which can therefore be projectively generated by the R_{p-2} of each series. This is possible only when the equation of one quadric on which $c_{2p-2}^{(p-1)}$ lies can be reduced to contain but four variables. If this hypercone be projected from its double R_{p-4} into R_3, $c_{2p-2}^{(p-1)}$ will project into $c_{2p-2}^{(3)}$ lying on a quadric surface.

The curve is of type $(p-1, p-1)$ and has $(p-1)(p-4)$ double points. *A quadratic identity between four adjoint curves can be found (by means of the equation of the curve itself) for every non-hyperelliptic curve of genus* $p > 3$.*

7. The linear transformations which leave $c_6^{(2)}$ invariant must also permute the double points among themselves. If

$$A_1 \equiv (1, 0, 0), \quad A_2 \equiv (0, 1, 0), \quad A_3 \equiv (0, 0, 1), \quad A_4 \equiv (1, 1, 1),$$

all the possible linear transformations are contained in the g_{24} formed by the $4!$ permutations of these points. As generating operations we may take the three harmonic homologies

$$(A_1 A_2)(A_3)(A_4) = (x_1 x_2)(x_3 x_4)(x_5 x_6),$$
$$(A_1 A_3)(A_2)(A_4) = (x_1 x_6)(x_2 x_4)(x_3 x_5),$$
$$(A_1 A_4)(A_2)(A_3) = (x_1 x_3)\begin{pmatrix} x^2 & x_4 & x_5 & x_6 \\ x_3 - x_1 + x_2 & -x_1 + x_2 - x_4 & -x_3 + x_1 + x_5 & x_5 - x_3 - x_6 \end{pmatrix}.$$

Since the quadratic identities (1) which are independent of the c_6 simply permute among themselves, in order to obtain the most general c_6 of $p = 6$ which is invariant under any group contained in the above octahedron group, simply write a general quadratic relation among the x_i and impose such conditions as will leave its form unaltered when operated upon by the generators of the group.

The only other operation which can leave the curve invariant is the quadratic

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*This theorem does not contradict that stated by Noether, *Math. Ann.*, Vol. XVII, p. 441. There the basis points a_1, a_2, ... are chosen arbitrarily.

inversion, having any three of the double points for basis points. That determined by $A_1 A_2 A_3$ and leaving the point A_4 fixed can be expressed in the form

$$(x_1 x_6)(x_2 x_5)(x_3 x_4).$$

The pencil of conics passing through all four double points defines a fifth linear series g_4^1, and may be denoted by A_5. The operation of inversion as to $A_1 A_2 A_3$ changes the pencil of straight lines through A_4 into A_5, and conversely. Hence

$$(A_4 A_5)(A_1)(A_2)(A_3) = (x_1 x_6)(x_2 x_5)(x_3 x_4).$$

These four generators will define the symmetric group of order 120, and proper combinations of them will define any group contained within it.

The largest period of any birational transformation which leaves a c_6 of type (a) *invariant is six.* These operations and the corresponding equations can now be immediately written down. In particular, if the curve belongs to the group generated by $(A_1 A_2)$, $(A_1 A_3)$, $(A_4 A_5)$, its equation is of the form

$$A \sum_{i=1}^{6} x_i^2 + B(x_1 x_2 + x_1 x_3 + x_2 x_4 + x_3 x_5 + x_4 x_6 + x_5 x_6) + C(x_1 x_6 + x_2 x_5 + x_3 x_4) = 0,$$

when proper use is made of equations (1). If it be invariant under $(A_1 A_4)$ also, $A = -2$, $B = 2$, $C = -1$. Expressed in terms of x, y, z this equation defines the c_6 having the maximum group of order 120. Its form is

$$2 \Sigma x^4 y z + 2 \Sigma x^3 y^3 - 2 \Sigma x^4 y^2 + \Sigma x^3 y^2 z - 6 x^2 y^2 z^2 = 0.$$

§ 2 (b). *Three Double Points Collinear.*

8. If this form be inverted as to a triangle of nodes, the third node on one of the sides of the triangle becomes a tacnode, hence the latter configuration is as general as the former. Let the tacnode be at (0, 0, 1), $x + a y = 0$ the equation of the tacnodal tangent. From the system of adjoint cubics we may write

$$\rho x_1 = x^2 y,\ \rho x_2 = x y^2,\ \rho x_3 = x^2 z,\ \rho x_4 = x z^2 + a y z^2,\ \rho x_5 = y^2 z,\ \rho x_6 = x y z,$$

from which the quadratic relations

$$\begin{gathered} x_1 x_6 - x_2 x_3 = 0, \quad x_1 x_5 - x_2 x_6 = 0, \quad x_3 x_5 - x_6^2 = 0, \\ x_2 x_4 - x_6(x_6 + a x_5) = 0, \quad x_1 x_4 - a x_3^2 + x_3 x_6 = 0 \end{gathered}$$

at once follow. Any curve c_6 having this configuration of nodes is completely defined by $\Sigma a_{ik} x_i x_k = 0$.

The only linear transformations that will leave this configuration invariant are of the form

$$\rho x_1^1 = x_1, \quad \rho x_2^1 = x_2, \quad \rho x_3^1 = k x_3, \quad \rho x_4^1 = k^2 x_4, \quad \rho x_5^1 = k x_5, \quad \rho x_6^1 = k x_6,$$

or of the form

$$\rho x_1^1 = x_2, \quad \rho x_2^1 = x_1, \quad \rho x_3^1 = x_5, \quad \rho x_4^1 = x_4, \quad \rho x_5^1 = x_3, \quad \rho x_6^1 = x_6.$$

In the first case $k^2 = 1$ or $k^4 = 1$, but the latter is possible only for $x_4^2 = x_1 x_2$, which is hyperelliptic, $p = 2$. For any odd value of k, c_6 is composite. By letting $a = 1$, which is no restriction, the curve having the non-cyclic G_4 becomes

$$a_1 x^3 y^3 + a_2 x^2 y^2 (x^2 + y^2) + a_3 z^4 (x + y)^2 + a_4 x^2 y^2 z^2 + a_5 (x^4 z^2 + y^4 z^2) + a_6 z^2 (x^2 + y^2)^2 = 0.$$

It was shown that no inversion can leave either type invariant, hence: *The most general birational group which leaves a c_6 of type* (b) *invariant is the non-cyclic linear G_4.*

If $a = 0$, the form may be written $\Sigma a_i x_i^2 = 0$.

The case of two tacnodes, and in particular, of one tacnodal tacnode passing through the other tacnode, can all have the non-cyclic G_4. The case of three consecutive collinear nodes and that of the oscnode are equivalent. The former is invariant under a harmonic homology, the latter under an inversion with coincident fundamental points. Moreover, both forms are invariant under one other harmonic homology, commutative with the first operation; hence these have also the same g_4.

Finally, if all four double points are consecutive, they must lie on a conic. If (0, 0, 1) be the point, $y = 0$ the tangent and $zy = x^2$ the conic on which the nodes lie, the system of adjoint ϕ_3 may be written in the form

$$\rho x_1 = y z^2 + x^2 z, \quad \rho x_2 = x y z + x^2, \quad \rho x_3 = y^2 z, \quad \rho x_4 = x^2 y, \quad \rho x_5 = x y^2, \quad \rho x_6 = y^3,$$

and the quadratic relations become

$$x_2 x_5 = x_3 x_4 + x_4^2, \quad x_2 x_6 = x_3 x_5 + x_4 x_5, \quad x_4 x_6 = x_5^2,$$
$$x_1 x_5 = x_2 x_3, \quad x_1 x_6 = x_3^2 + x_3 x_4, \quad \Sigma a_{ik} x_i x_k = 0.$$

The only operation which will leave these forms invariant is changing the signs of x_2 and x_5. It is a harmonic homology with center on the tangent and axis through the node.

9. It will be noticed that in all forms under (b) at least one of the quadratic identities involved but three of the variables. In each case $\Sigma a_{ik} x_i x_k = 0$ may be particularized to include many more such forms, even when the form of the nodes is prescribed.

Any quadratic relation involving but three adjoint ϕ may be written in the form

$$2\phi\phi'' - \phi'^2 = 0,$$

which, with the other quadratic relations, defines c_6 without extraneous factors; at every variable point in which ϕ, ϕ' intersect, the former touches c_6. Similarly for ϕ'', ϕ'. If the curve $a\phi + b\phi' + c\phi'' = 0$ cuts c_6 in the two sets $(\phi_1, \phi_1', \phi_1'')$, $(\phi_2, \phi_2', \phi_2'')$, then c_6 may also be written in the form

$$2(\phi\phi_1'' + \phi_1\phi'' - \phi'\phi_1')(\phi\phi_2'' + \phi_2\phi'' - \phi'\phi_2') = (a\phi + b\phi' + c\phi'')^2;$$

hence if there be two contact curves, there will be an infinite number. These systems are the images of the tangent R_4 to the cone $2\phi\phi'' - \phi'^2 = 0$ having an R_2 for vertex. Since through a set of $p - 1 = 5$ points of contact both ϕ and ϕ' pass, they are the basis points of a pencil. In general, if l be any line and ψ_{n-2} be a curve of order $n - 2$ passing through the nodes, a special group of $p - 1$ points and $n - 1$ of the intersections of c_n, l, then the net

$$a l\phi + b l\phi' + c\psi = 0$$

will have $p - 1 + n - 3$ fixed basis points in addition to the nodes; this leaves $p + 2$ variable intersections. This net can now be used to transform c_n birationally into another of order $p + 2$. The point $(0, 0, 1)$ is a triple point P_3 on c_{p+2}, since the pencil of straight lines through it corresponds to $a\phi + b\phi' = 0$. A contact curve must go into a contact curve; hence some line of the pencil $ax_1 + bx_2 = 0$, counted twice, is image of the contact curve. It is a factor of an adjoint curve; the remaining nodes lie on a curve of order $p - 3$. Now let a_{p-1} be a curve passing through the triple point and all the double points but one, b_{p-2} a curve passing through all the double points, $k \equiv x_1 + kx_2 = 0$ be a line passing through the triple point and the node not lying on a_{p-1}. The net formed by $x_1 b$, $x_2 b$, a can now be used to transform c_{p+2}. The transformed curve will be of order $p + 1$, and the pencil through $(0, 0, 1)$ remains invariant; as before, one of its lines must count double. The remaining double points lie on a curve of order $p - 4$, but this is impossible unless the special line contains a second double point, hence $(0, 0, 1)$ is a tacnode. For $p = 4$ and $p = 5$ the curve can not be reduced to a simpler form, but for $p > 5$, it is always possible to further reduce the order of the curve.

10. These steps can be easily interpreted geometrically. The ϕ represents an R_4, tangent to the quadric cone of R_5, having an R_2 for vertex. Each tangent R_4 touches $c_{10}^{(6)}$ in five points, the points of contact lying in an R_3. Let A, B, C, D, E be the points of contact. Project $c_{10}^{(6)}$ from A into an R_4 not passing

through A. The $c_9^{(4)}$ contains the images A', B', C', D', E'; it will be touched by an R_3 in B', C', D', E' which also passes simply through A'. The four points of contact lie in an R_2. Now project $c_9^{(4)}$ from B' into R_3 not passing through B'. An R_2 touches $c_8^{(3)}$ in C'', D'', E'' and passes through A'', B''. The points of contact are collinear. Project $c_8^{(3)}$ from C'' into R_2 not passing through C''. The $c_7^{(2)}$ will have a tacnode, and the images of the other points from which the successive curves were projected are the residual intersections of the tacnodal tangent and the curve. The g_7^2 formed by the lines of the plane of c_7 is such that if A'' be adjoined to each group, the series g_8^2 is incomplete, being contained in a g_8^3, similarly for g_9^4, g_{10}^5. If we construct a system of $\infty^5 \phi_4$ such that when two further basis points (B''', E''') are given, a g_8^3 will be defined, and further such that the g_8^2 obtained by fixing one more basis point will have as partial series the straight lines of the plane, it is only possible when the seven remaining double points lie on a conic.*

11. If $c_{10}^{(5)}$ be projected into R_2 from the vertex R_2 of the quadric cone, the result is a conic, counted five times. If $c_{2p-2}^{(p-1)}$ be projected from an R_{p-4} vertex of a quadric cone on which the curve lies into R_3, the resulting conical curve will cut each generator in $p-1$ points and have $(p-1)(p-4)$ actual double points. If $c_{2p-2}^{(3)}$ be projected into R_2 from one of these double points, the $c_{2p-4}^{(2)}$ will have $p-3$ branches touching each other at a common point, and $(p-1)(p-4)-1$ other double points lying on ϕ_{p-3}. Both this form and the preceding one can be obtained without the use of special groups.

12. Now suppose there are two quadratic relations which involve but three variables. Through every point of $c_{10}^{(5)}$ now pass two tangent R_4, each of which touches $c_{10}^{(5)}$ in four other points. In the two correspondences formed by the tangent R_4, it will happen for a finite number of points that the two R_4 will also have another point of $c_{10}^{(5)}$ in common. Now proceed as before, first projecting from one of these points, then from the other. The $c_8^{(3)}$ has an actual double point, through which pass two planes, each of which touches $c_8^{(3)}$ in three other points, the points of contact being collinear.

§ 3 (c). c_6 *has a* g_3^1.

13. When a curve of genus 6 and having a g_3^1 is reduced to c_6, the curve must have a triple point.†

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* This same result was obtained by Kraus, *Math. Ann.*, Vol. XVI. by a partly different method.

† Amodeo, "Curve k-gonali," *Ann. di Mat.* (2), Vol. XXI (1893), p. 221.

If the triple point be chosen at (0, 0, 1) and the double point at (0, 1, 0), the system of adjoint ϕ_3 may be written

$$\rho x_1 = x^2 z, \quad \rho x_2 = xyz, \quad \rho x_3 = y^2 z, \quad \rho x_4 = x^3, \quad \rho x_5 = x^2 y, \quad \rho x_6 = xy^2,$$

from which

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_4}{x_5} = \frac{x_5}{x_6},$$

defining six linearly independent quadratic relations. This system defines a rational ruled surface of order 4, common to all the six quadrics, which are therefore not sufficient to define the curve.*

On the other hand, it is not difficult to discuss these curves directly from their equations in the plane. The general form is

$$f_3(x, y) z^3 + f_4(x, y) z^2 + \psi_4(x, y) xz + \psi_3(x, y) x^2 y = 0.$$

If $\psi_3(x, y) = f_3(y, x)$ and $\psi_4(x, y) = f_4(y, x)$, c_6 is invariant under $\rho x' = yz$, $\rho y' = zx$, $\rho z' = xy$. If in addition $f_4 \equiv 0$, c_6 has the cyclic perspectivity

$$\sigma x' = x, \quad \sigma y' = y, \quad \sigma z' = \omega z, \quad \omega^3 = 1.$$

The latter can exist alone if $f_4 \equiv 0$, $\psi_4 \equiv 0$.

In particular, the curve

$$x^2 y (ax^3 + by^3) + z^3 (bx^3 + ay^3) = 0$$

has the quadratic inversion and

$$\rho x' = \theta^4 x, \quad \rho y' = \theta y, \quad \rho z' = z, \quad \theta^9 = 1,$$

defining the dihedral G_{18}. The forms having a G_4 generated by a harmonic homology about x or y and the inversion can be immediately written down.

The curve

$$x^3 z^3 + (ax^4 + by^4) z^2 + (cx^4 + dy^4) xz + kx^2 y^4 = 0$$

has the cyclic G_4 defined by $\begin{pmatrix} x & y & z \\ x & iy & z \end{pmatrix}$. In particular, if $c = b$, $d = a$, $k = 1$, it also admits the quadric inversion, thus defining a dihedral G_8. The point (0, 0, 1) has $x = 0$ for triple tangent; at the double point (0, 1, 0) each tangent has five-point contact. The line $y = 0$ meets c_6 in three other points, at each of which the tangent has four-point contact and passes through the double point. The curve has 32 other points of inflexion, arranged on eight lines passing through the double point.

Of the two forms having four coincident double points, that with a simple branch passing through a tacnode may have at most a single harmonic homology, as

$$axy^2 z^3 + by^4 z^2 + y^2 \phi_2(x^2, y^2) + xy^2 z (cx^2 + dy^2) + cx^5 z + fx^2 y^2 z^2 = 0.$$

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*Kraus, *l. c.*; Snyder, "On Birational Transformations of Curves of High Genus," JOURNAL, Vol. XXX 1908), p. 10.

That with a simple branch passing through a cusp of the second kind,

$$zy(x-ay^2)^2+bx^2y^2z^2+x^2yzf_2(x,y)+cx^3yz^2-dx^4z^2+x^2\phi_4(x,y)=0,$$

has no invariant transformations.

§ 4 (d). *The Non-singular Quintic.*

14. It has been shown* that if a curve of genus 6 has a g_5^2 it could not be reduced to a sextic. The non-singular curves have at most only linear transformations into themselves. The forms of the possible linear groups to which c_5 can belong have already been determined.†

The adjoint curves are made up of all the conics of the plane. If we write

$$\rho x_1=x^2,\ \rho x_2=xy,\ \rho x_3=y^2,\ \rho x_4=xz,\ \rho x_5=yz,\ \rho x_6=z^2,$$

then

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_4}{x_5},\ \frac{x_1}{x_4}=\frac{x_4}{x_6}=\frac{x_2}{x_5},\ \frac{x_2}{x_4}=\frac{x_3}{x_5}=\frac{x_5}{x_6},$$

or

$$x_1x_3=x_2^2,\ x_1x_5=x_2x_4,\ x_2x_5=x_3x_4,\ x_1x_6=x_4^2,\ x_4x_5=x_2x_6,\ x_3x_6=x_5^2.$$

Hence, here too, the six quadratic relations are independent of the quintic curve. This is the only case thus far discovered of a curve not having a g_3^1 which is not defined by the quadratic relations among the adjoint curves. The six quadrics have a surface in common, but not a ruled surface. It is the Veronese surface of order 4. It can be projected from $(0, 0, 0, 0, 0, 1)$ into $x_6=0$ as the rational ruled surface of order 3,

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_4}{x_5},$$

and therefore, from the preceding section, it also follows that *if the normal curve be projected from a bisecant, it projects into a conic, counted four times.* We have three interesting projections into R_3. If the surface be projected from any bisecant, the result is a quadric surface. If the line have but one point in common with the surface in R_5: the result is a ruled cubic of the first kind, as

$$x_3x_4^2=x_1x_5^2.$$

Finally, by projecting from a line not having any point on the surface, we obtain, for example,

$$\sqrt{x_1+x_3+x_6+2(x_2+x_4+x_5)}=\sqrt{x_1}+\sqrt{x_3}+\sqrt{x_6},$$

a Steiner surface.‡

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* Snyder, *l. c.*

† Snyder, "Plane Quintic Curves Which Possess a Group of Linear Transformations," JOURNAL, Vol. XXX (1908), p. 1. The most interesting type is $x^5+y^5+z^5=0$, which is invariant under a group of order 150.

‡ An excellent discussion of the Veronese surface is given by Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi*, Pisa, 1907. See Chapter XV.

§ 5 (e). *Hyperelliptic Curves.*

15. The canonical form of a hyperelliptic curve of genus 6 is

$$y^2 z^{12} = f_{14}(x, z).$$

The characteristic G_2 is the homology

$$\begin{pmatrix} x & y & z \\ x & -y & z \end{pmatrix} = H.$$

If $f_{14}(x, z) = \phi_7(x^2, z^2)$, we have the non-cyclic G_4. If $f_{14}(x, z) = f_{14}(z, x)$, another G_4, defined by H and

$$K \equiv \begin{pmatrix} x & y & z \\ x^6 z & y z^6 & x^7 \end{pmatrix}.$$

The dihedral G_8 arises when $\phi_7(x^2, z^2) = \phi_7(z^2, x^2)$.

By inversion, the equation of the curve may be written

$$y^2 z^{11} = f_{13}(x, z).$$

If $f_{13}(x, z) = x\,\phi_6(x^2, z^2)$, we have the cyclic

$$G_4 \equiv \begin{pmatrix} x & y & z \\ -x & iy & z \end{pmatrix}.$$

If $\phi_6(x^2, z^2) = \phi_6(z^2, x^2)$, the curve also admits k, making a dihedral G_8.

$y^2 z^{11} = x f_4(x^3, z^3)$ has $\begin{pmatrix} x & y & z \\ \theta^2 x & \theta y & z \end{pmatrix}$, $\theta^6 = 1$; if $f_4(x^3, z^3) = f_4(z^3, x^3)$, the dihedral G_{12}; $y^2 z^{11} = x f_3(x^4, z^4)$, the cyclic $G_8 \equiv \begin{pmatrix} x & y & z \\ ix & \sqrt{i}\,y & z \end{pmatrix}$; and if f_3 is symmetric, the dihedral G_{16}. In particular, $y^2 z^{11} = x(x^4 + z^4)(x^8 - 14 x^4 z^4 + z^8)$ has a G_{48}, formed by H and the octahedron group.

$y^2 z^{11} = x(x^{12} + z^{12})$ has the dihedral G_{48}.

$y^2 z^{11} = x f_2(x^6, z^6)$ has the cyclic $G_{12} \equiv \begin{pmatrix} x & y & z \\ \theta x & \sqrt{\theta}\,y & z \end{pmatrix}$ and, if f_2 is symmetric, the dihedral G_{24}; $y^2 z^{11} = x^{13} + z^{13}$, the cyclic G_{26}. This is the only operation of period as high as 26 that any curve of genus 6 can have.

$y^2 z^{12} = x^{14} + z^{14}$ has the dihedral G_{28}, and H, making a group of order 56.*

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* A. Wiman, "Ueber die hyperelliptischen Curven und diejenigen vom Geschlecht $p=3$, welche eindeutige Transformationen in sich zulassen," *Bihang t. k. Svenska Vetensk. Akad. Handlingar*, Band XXI (1895).

CORNELL UNIVERSITY, *January*, 1908.

On the Range of Birational Transformation of Curves of Genus Greater than the Canonical Form.

By Virgil Snyder.

In a former paper[*] I have shown that no curve of order n can be birationally transformed into itself or other curve of order n, if it have fewer than $E\left(\frac{n-1}{2}\right)^2-2$ double points, $E(k)$ being the largest integer less than k. In the present paper I show what is the minimum order of the transformed curve, determine the nature of the transformation itself, and show how certain curves of this type can be generated.

1. If a non-singular curve c_n of order n be transformed birationally into c_y by means of adjoint ϕ_x, the minimum value of y is obtained when all the basis points of a net $(\infty^2)\,\phi_x$ are taken upon c_n. This number is x^2-x+1;[†] hence

$$y=nx-x^2+x-1.$$

Since we exclude collineations, and are concerned with special series g_y^2 only,

$$1<x\leqq n-3.$$

Under these conditions y reaches its minimum value $2n-3$ when $x=2$. This requires that the net of transforming curves be a system of ∞^2 conics circumscribing a triangle whose vertices lie on c_n; thus the transformation is a quadratic inversion. Hence:

The curve of lowest order into which a non-singular curve of order n *can be transformed by birational transformation other than collineation is of order* $2n-3$, *and the transformation is birational for the entire plane.*

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[*] Journal, Vol. XXX (1908), pp. 10–18.

[†] C. Küpper: "Ueber das Vorkommen von linearen Schaaren. . . . ,' *Sitzungsberichte der Böhmischen Gesellschaft. . . .*, Prag, 1892, pp. 264–272.

But by inversion, the $n-2$ points on each side of the triangle will go into the opposite vertices; hence:

The necessary and sufficient condition that a curve of order $2n-3$ and genus $\frac{1}{2}(n-1)(n-2)$ be birationally transformable into a curve of order n is that it have three multiple points of order $n-2$.

Incidentally, no curve of this nature can also have a linear series g^2_k, k being any integer between n and $2n-3$.

2. This result points out a curious exception to the canonical form of curves of genus p* when $p=6$. The general theorem is that any curve of genus 6 can be reduced to a sextic with four double points, but this is not true of a non-singular quintic, as the simplest curve to which it can be transformed is a curve of order 7, having three triple points. This is the only exception to the general theorem for any genus. *No curve of genus 6 can have a linear g^2_5 and a linear g^2_6, but every such curve has one or the other series.* A c_7 with three triple points is not birationally equivalent to a c_7 of genus 6 with any other configuration of multiple points. Every curve of genus 6 can be transformed to a c_7 without the use of special groups (Clebsch-Lindemann, *l. c.*, p. 689).

3. The same value of y that was determined for $x=2$ is also obtained for $x=n-1$, but this case does not need to be considered, since the special groups can always be defined by simpler curves. However, as an illustration of a net of curves having the maximum number of basis points on a given one, the following curve will be of interest. Consider the curve

$$xy^n + yz^n + zx^n = 0$$

and the linear transformations

$$S\begin{cases} x' = x \\ y' = \theta y \\ z' = \theta^n z \end{cases}, \quad \theta^{n^2-n+1} = 1. \quad T\begin{cases} x' = y \\ y' = z \\ z' = x \end{cases}.$$

The curve is invariant under the group generated by S and T. Since $S^{n-1}T = TS$, the group is of order $3(n^2-n+1)$, its operators being of the form S^k, S^lT, S^mT^2. S^k is of order n^2-n+1, the others being of order 3.

The curve is non-singular, $p=\frac{1}{2}n(n-1)$, and the order of S is $2p+1$. The coordinate triangle is invariant under the group. Its sides have n-point contact with the curve at one vertex and a simple intersection at the other.

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* Clebsch-Lindemann: *Vorlesungen über Geometrie*, Vol. I, p. 709. Hyperelliptic curves are excluded.

This accounts for $3(n-2)$ points of inflexion and $\frac{3}{2}(n-2)(n-3)$ double tangents. The remaining $3(n^2-n+1)$ points of inflexion are arranged in three sets of n^2-n+1 each, invariant under S, and also in n^2-n+1 triads invariant under T, one of which is real. From this configuration it follows that if $n>3$, the given curve can not have a larger group than that generated by S and T. If $n=3$ all the inflexions are ordinary; the c_4 is now invariant under the simple group of order 168.

The invariant points of T are $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, $\omega^3=1$. The line joining the imaginary points, $x+y+z=0$, is a bitangent when n is a multiple of 3, the points of contact being the invariant points. The number of bitangents apart from the invariant triangle is $\frac{1}{2}(n^2-n+1)(n^2+3n-10)$. When n is a multiple of 3, this number is not a multiple of 3, but congruent 1, the invariant bitangent under T. The inflexions and bitangents can be curiously arranged on conics of the form

$$(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)-k(x+y+z)^2=0,$$

and n^2-n other systems into which this is transformed by S. But the most important configuration for our purpose is that formed by any point P, and its images under S. We shall first prove the following theorem:

Through the n^2-n+1 points of any cycle of S can be passed ∞^2 curves of order n.

Consider the general equation of a c_n written down with unknown coefficients. It defines $\infty^{\frac{n}{2}(n+3)}$ curves. After passing through n^2-n+1 points, we still require two arbitrary constants, which necessitates that all the determinants of order $\frac{n}{2}(n+3)-1$ in the matrix formed by n^2-n+1 rows and $\frac{n}{2}(n+3)+1$ columns should vanish. This is also a sufficient condition. The successive images of (a, b, c) are $(a, b\theta, c\theta^n), \ldots, (a, b\theta^{n^2-n}, c\theta^{-n})$. When the coordinates of these points are substituted in the equation of c_n it will be seen that the following pairs of coefficients only differ by a constant:

$$x^{n-1}z,\ y^n; \quad x^n,\ yz^{n-1}; \quad xy^{n-1},\ z^n;$$

hence not only all the determinants of the matrix vanish, but also all the first and second minors, for at least two columns of each second minor will always be equal. This proves the proposition. Moreover it also follows that not every minor of the third order vanishes.

If this net of c_n be used as transforming curves, the original c_{n+1} is transformed into a curve of order $2n-1$. According to our theorem, the new curve must have three points of multiplicity $n-1$; hence the transforming curves must define three linear series g^1_n, which requires that all the basis points of three pencils must lie on c_{n+1}. These pencils are determined by the n^2-n+1 images of (a, b, c) on c_{n+1} and any vertex of the invariant triangle. For example, the pencil belonging to $(1, 0, 0)$ is

$$ab^{n-1}z^n - c^n xy^{n-1} + k(a^{n-1}cy^n - b^n zx^{n-1}) = 0.$$

The curves of any pencil in the net must have the remaining basis points on a straight line. When the point fixing the pencil is an invariant point, the curves have $(n-1)$-point contact with a side of the triangle. The equations of the transformation may then be written

$$\rho x' = ab^{n-1}z^n - c^n xy^{n-1}, \quad \rho y' = bc^{n-1}x^n - a^n yz^{n-1}, \quad \rho z' = a^{n-1}cy^n - b^n zx^{n-1},$$

from which

$$x'y + y'z + z'x = 0.$$

From these equations, the equation of c_{n+1} and the condition that (a, b, c) is a point upon it, we obtain

$$\sigma x' = acxz, \quad \sigma y' = abxy, \quad \sigma z' = bcyz.$$

The original c_{n+1} can be generated by the pencil

$$ab^{n-1}z^n - c^n xy^{n-1} + \lambda(bc^{n-1}x^n - a^n yz^{n-1}) = 0$$

and the projective pencil $cz + \lambda by = 0$; hence the groups of g^1_n lie on straight lines passing through the invariant point opposite to the $(n-1)$-point tangent, independently of the point (a, b, c). This completes the reduction of the transformation to the Cremona type.

4. Now suppose c_n has δ distinct double points. In this case

$$y = nx - x^2 + x - 1 - \delta, \qquad \delta < E\left(\frac{n-1}{2}\right)^2,$$

since otherwise y would certainly not be greater than n; this case was considered in my former paper.

If $\delta = 1, 2, 3$, the preceding argument will apply directly; the new curve is of order $2n-4$, $2n-5$, $2n-6$, respectively, and can be obtained by inversion. Since the δ points are assumed to be distinct, the lowest value of y that can be obtained by inversion is $2n-6$. Further, if $\delta \leqq 2(n-4)$, by no other transformation can c_n be reduced to a curve of order as low as $2n-6$, when $n > 8$.

5. For lower values of n, the various cases can be disposed of separately. If $p = 5$ and c_n has g^2_5, it must also have a g^1_3 by the Riemann-Roch theorem; hence the standard form of c_6 is one with a triple point. Since $p = 2 \cdot 5 - 5$, the lines joining the triads of g^1_3 must all pass through a common point. *A sextic curve with a triple point and two double points can not be birationally transformed into a sextic with any other configuration of multiple points.*

If $p = 7$, we can at once say: Any curve of genus 7 can be reduced to c_7 with 8 double points. If these be distinct it can not be further reduced. If c_7 has $2P_3 + 2P_2$, it also has g^2_6 and can be reduced to a c_6 with 3 double points at the vertices of a triangle. If the $2P_2$ be replaced by a tacnode, c_6 has three collinear double points. If c_7 has $P_4 + 2P_2$, c_6 has P_3. These three forms are birationally distinct.

For $p = 8$, the canonical series is g^2_8. If g^2_6 exists, g^2_7 must also, but not conversely. If the 13 double points of c_8 are distinct, the c_8 can not have either. Let c_4, c'_4 be two quartics intersecting in three points on a given straight line c_1. Through the 13 residual points of intersection, and any four points on c_1 pass a pencil of quintics $c_5 + \lambda c'_5$. Make the two pencils projective in such a way that corresponding curves will intersect on c_1. The locus of all the intersections will be a c_9, having c_1 as factor. The resulting c_8 will have at least 13 double points, through which pass a net of quartics, cutting a g^2_8 on c_8, but they can not be used to transform the curve.*

Conversely, a c_6 with two double points can not be birationally transformed into a curve of order 8 with 13 distinct double points. When a binodal c_6 is transformed into c_7, the latter has two triple points.

For $p = 9$, g^2_6 and g^2_7 are mutually exclusive. A $c_8^{(9)}$ having g^2_6 must have $P_4 + 2P_3$, but a c_7 with 6 double points can be transformed into c_8 with 12 double points by adjoint cubics. The c_8 has the property that a net of adjoint quintics can be passed through the 12 nodes and 9 simple points on the curve. Such a curve can be easily constructed by the above method. When a c_8 of genus 9 is the projection of a space curve of order 9, it can be reduced to a c_7, since when $p = 9$, g^2_7 and g^3_9 are reciprocal series by the Riemann-Roch theorem. Conversely, from every c_7 with 6 distinct double points we can define a g^3_9 by means of adjoint ϕ_3; hence when a cubic and a quartic surface intersect in a space cubic curve, the residual intersection is a space curve of order 9, having 19 apparent double

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*See Snyder: "On a Special Net of Algebraic Curves," *Bull. Amer. Math. Society*, Vol. XIV (1907), p. 70.

points. Through the 19 bisecants from an arbitrary point can be passed a net of quintic cones.

The larger values of p offer no exception to the general case.

6. If the δ distinct double points be replaced by s_i-fold points such that $\frac{1}{2}\Sigma s_i(s_i-1) \leqq 2(n-4)$, the orders of the transformed curves will be lower than $2n-6$, but, as before, the curve of lowest order can be obtained by inversion, the three multiple points of highest order which are not collinear being the basis points. If $\frac{1}{2}\Sigma s_i(s_i-1) = 2(n-4)$, and ϕ_x has an (s_i-1)-fold point at each s_i-fold point of c_n, then $\frac{1}{2}\Sigma s_i(s_i-1)$ conditions are imposed upon ϕ_x and $\Sigma s_i(s_i-1)$ intersections with c_n are provided for. If now we assume the extreme case of x^2-1 basis points, then

$$y = nx - \Sigma s_i(s_i-1) - \left\{x^2 - 1 - \Sigma \frac{s_i}{2}(s_i-1)\right\} = nx - 2(n-4) - x^2 + 1.$$

When $x = n-3$, $y = n$, but this is only possible when the sum of the three highest s_i is greater than $n-3$, in which case the number of double points would be greater than $2(n-4)$. In every case y is greater than $2n-s_1-s_2-s_3$; hence:

The curve of lowest order into which a curve of order $m > 8$ *and genus* $p \geqq \frac{1}{2}(m-1)(m-2) - 2(m-4)$ *can be birationally transformed can be obtained by quadratic inversion.*

7. It is shown in the theory of space curves that every algebraic space curve can be represented by a cone k_m of the same order as the curve, and the monoid $w = \frac{k_{n+1}}{k_n}$, wherein k_n is a cone containing all the double edges of k_m, and k_{n+1} passes through all the intersections of k_n, k_m. When the curve is given as the complete or partial intersection of two surfaces, the equation of k_m is obtained by eliminating one of the variables (unless the given curve is a conical curve) and the monoid appears incidentally in the process of elimination.*

8. Consider the twisted curve

$$w^{n+1} + a_1 w^n + a_2 w^{n-1} + \ldots\ldots = 0, \quad w^2 + b_1 w + b_2 = 0,$$

wherein a_i, b_i are ternary forms of order i. When the intersection is complete, it will be of symbol $(n+1, n+1)$, or say (n, n). If partial, of symbol $(n, n+1)$.

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* See Cayley: "On Halphen's Characteristic n....," *Crelle*, Vol. CXI (1893), pp. 347–352.

In the latter case the curve is of order $2n+1$, and has n^2 apparent double points. The n^2 bisecants from any point in space are the basis-edges of a pencil of cones of order n. If the plane projection be c_{2n+1}, and ϕ_n, ϕ'_n be two adjoint curves of order n, then

$$\phi_n \cdot c_1 + c'_1 \cdot \phi'_n = 0$$

will define $\infty^5 \, \phi_{n+1}$ and in general

$$\phi_n \cdot c_r + \phi'_n \cdot c'_r = 0$$

will define $(r^2 + 3r - 1)$-fold ϕ_{n+r}, wherein

$$1 \overset{=}{<} r \overset{=}{<} n - 2.$$

Hence:

If a curve of order $n + r$ *can be passed through* $n^2 - \frac{1}{2}(n - r - 2)(n - r - 1)$ *of the* n^2 *double points of the projection of a space curve of order* $2n + 1$ *and genus* $n^2 - n$, *then it will pass through all* n^2 *double points.**

This curve can evidently be birationally transformed into a curve of order $2n$, since the space curve can be projected into such a curve from a point upon it. The question now arises, what is the lowest order to which such a plane curve can be birationally reduced?

The adjoint ϕ_{n+r} has still $r^2 + 3r + 1$ constants, and the number of variable intersections is $(2n+1)(n+r) - 2n^2$. If the order of the transformed curve is y, then all but y of these variable intersections must be fixed basis points, and the system of adjoint ϕ_{n+r} passing through them still have two arbitrary constants.

9. The question may now be stated thus: Given ∞^{r^2+3r+1} adjoint curves of order $n + r$, it is required to find $(2n + 1)(n + r) - 2n^2 - y$ fixed basis points upon c_{2n+1}, such that through them will pass ∞^2 curves of the system. This problem is formulated and solved in the Brill-Noether paper in Vol. VII of the *Math. Annalen* (§ 9, p. 290) for the case of a curve of general moduli. Thus, if we put

$$t = r^2 + 3r + 1, \quad R = (2n + 1)(n + r) - 2n^2 - y, \quad q = 2,$$

then (D) (p. 291 of the B.-N. paper),

$$R \geqq (q + 1)(R - t + q)$$

becomes

$$2y \geqq 4nr + 2n - 7r - 3r^2 + 3.$$

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* R. Sturm: "On Some New Theorems on Curves of Double Curvature," *British Association Report* (1881), p. 146. Sturm's theorem is more general than that here derived, but obtained in a different way. A different proof is given by Noether: "Zur Grundlegung der Theorie der algebraischen Raumcurven," *Berliner Abhandlungen*, 1883.

Since we need only consider values of r within the interval $1 \leqq r \leqq n-2$, the minimum value of y is $3n-2$. Hence the method of counting the conditions will be of no service, as it presupposes that c_{2n+1}, $p=n^2-n$, is a general curve of its class, while our curve is a highly particularized one. For $n=1$ or $n=2$, the basis points may be arbitrary to reduce c_{2n+1} to c_{2n}. For $n=3$, we have c_7 with 9 double points. It is therefore possible to pass a net of ϕ_4 through these 9 double points (which lie on a pencil of cubics) and 4 other points on c_7, thus defining a g_6^2. The transformed c_6 must have a triple point and one double point in order to have a g_7^2.

10. Now let $P \equiv (0, 0, 0, 1)$ be any point not on the space curve R_{2n+1}. Call the cone from this point $k_{2n+1}(x, y, z) = 0$. The simplest monoid will be

$$w = \frac{f_{n+1}(x, y, z)}{f_n(x, y, z)}.$$

f_n passes through the n^2 bisecants from P, and has n other lines in common with k_{2n+1}. f_{n+1} passes through both the n^2 bisecants and the n simple lines common to the other two cones. Let Q be any residual point of intersection of f_{n+1}, k_{2n+1}. The ∞^2 planes through Q will cut R_{2n+1} in a linear g_{2n}^2. Project each of the sections of these planes and the monoid from P, and cut the cones with the plane $w=0$. These plane curves will pass through the n^2 double points of c_{2n+1}, through the n simple points and the projection of the point Q. They are therefore adjoint ϕ_{n+1}, have two degrees of freedom, and have the maximum number of basis points

$$n^2 + n + 1 = (n+1)^2 - (n+1) + 1$$

on c_{2n+1}. When this net is used to transform the curve into c_{2n}, the transforming curves go into the ∞^2 straight lines of the plane, *i. e.*, the space curve is projected from the point Q; hence c_{2n} has one n-fold point, and one $(n-1)$-fold point. Since by partial elimination of w between the equation of the quadric and any $F_{n+1}(x, y, z, w)$ containing R_{2n+1} we can obtain a series of monoids of order $n+r$ and a series of corresponding adjoint curves, ϕ_{n+r}, we say:

The plane projection of the space curve of symbol $(n, n+1)$ *on the quadric surface can be birationally transformed into a curve of order* $2n$ *by means of adjoint curves of order* $n+r$, $1 \leqq r \leqq n-2$, *having* $(n+r)(2n+1) - 2n^2 - 2n$ *simple fixed basis points on the given curve. The transformed curve will have one* n*-fold* *point and one $(n-1)$-fold point, and can not be birationally transformed into any simpler curve.*

11. If the curve be the complete intersection of a quadric and a surface of order n, $h=n^2-n$. A cone of order $n-1$ can be passed through the bisecants, and therefore ∞^3 cones of order n. If f_{n-1}, f_n be the lower and upper cones of the simplest monoid passing through the curve, this system of cones may be written

$$f_n + k_1 \cdot f_{n-1} = 0,$$

$k_1 = 0$ being any plane through the common vertex. This is the maximum number of basis points a triply infinite system of curves can have.*

The k_{n-1} passing through the n^2-n bisecants from P can have no further intersection with k_{2n}. The upper cone f_n will pass through the bisecants, and $2n$ simple edges of the lower cone. The ∞^2 planes through Q will cut the monoid in curves of order n which are projected into adjoint ϕ_n in $w=0$. They all pass through the image of the point Q; hence, as before, a net of ϕ_n have n^2-n+1 basis points on c_{2n}. The transformed curve is of order $2n-1$ and is obtained by projecting the twisted curve from Q. Any one of the ∞^3 projecting cones can be taken as superior cone of the minimum monoid.

12. The procedure is now easily generalized. Given any space curve $R_m=0$ defined by the cone $k_m=0$ and the monoid

$$w = \frac{f_{n+1}}{f_n}.$$

Since f_n passes through all the bisecants of R_m from P and f_{n+1} through the complete intersection k_m, f_n, and since through this intersection ∞^3 cones of order $n+1$ pass, hence a net of adjoint curves of order $n+1$ always exists which will transform the curve into another, of order $m-1$. For no other curve than those of symbols (n, n), $(n, n+1)$ on the hyperboloid is the maximum number of basis points employed. Let (a, b), $a \leqq b$ be the symbol of a general R_{a+b} on a hyperboloid. The inferior cone of the monoid is f_{b-1}, and the number of basis points for $\infty^2 f_b$ is

$$\frac{a(a-1)+b(b-1)}{2} + a(b-a) + 1.$$

When $a=b$ or $a=b-1$, this number is b^2-b+1. For other values of a it is smaller.

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* C. Küpper: "Bestimmung der Minimalbasis ....," *Monatshefte der Math. und Physik*, Vol. VI (1895), pp. 5–11.

13. The same reasoning will apply directly to space curves which are the complete intersections of two surfaces, F_μ, $F_{\mu'}$. Here

$$m = \mu\mu', \quad n = (\mu - 1)(\mu' - 1), \quad h = \frac{\mu\mu'}{2}(\mu - 1)(\mu' - 1).$$

The lower cone is fixed, and the upper one is fixed by the two conditions of passing through the bisecants, and residual intersection of $k_{\mu\mu'}$, k_n; hence we have but three degrees of freedom, just sufficient to reduce the plane curve to order $\mu\mu' - 1$. The transforming curves are of order $n + 1$. *The plane curves which are the projections of complete intersections of two surfaces of order μ, μ' can not be birationally reduced to order less than $\mu\mu' - 1$.*

As an interesting illustration, consider the two space curves of order 9, the complete intersection of two cubic surfaces, and the quadric curve of type (3, 6). In both cases $h = 18$. In the first, $n = 4$; in the second, $n = 5$. When projected from a point upon it, the first becomes a c_8 with 11 distinct double points; the latter a c_8 with a $P_4 + P_3$. Each can be transformed into another c_8 in an infinite number of ways, but neither can be transformed into any other type, or to a curve of lower order. The first transformation is made by means of ϕ_5, the second by quadratic inversion. The complete intersection of F_μ, $F_{\mu'}$ is projected from a point upon it into $c_{\mu\mu'-1}$, having $\frac{mn}{2} + 2 - m$ double points. Through them can always be passed ϕ_n, but not always a net. Thus if $\mu = \mu'$, the minimum curves of transformation are ϕ_{n+1}, and for $\mu' = 2$ they are conics, since the inferior cone of the monoid of a quadric curve, vertex on the curve, is the plane of the two generators through the vertex.

14. In general, the ∞^1 transformations of the projection of any R_m into c_{m-1} may be effected as follows: Let Q, S be any two points on the curve. By means of the sections of the monoid from any point P by the ∞^2 planes through Q we have already one such transformation. Similarly for the sections of the same monoid by the net of planes through S; hence these sections will define g^2_{m-1} on the c_{m-1} from P; but this can be more simply done by the sections of the monoid from P. The planes through SP will define a pencil of lines through the image of S. The others will project into ϕ_{x+1}, ϕ_x being the minimum cone through the trisecants passing through P.

CORNELL UNIVERSITY, *July*, 1907.

A Set of Assumptions for Projective Geometry.[*]

BY OSWALD VEBLEN AND JOHN WESLEY YOUNG.

CONTENTS.

PAGE.

Introduction. 347

§1. The Assumptions for General Projective Geometry. 348

§2. Algebra of Points and the Introduction of Analytic Methods. . 352

§3. Nets of Rationality. 356

§4. Assumption H and the Definition of Separation. 359

§5. The Assumption of Continuity and the Definition of a Chain. . 362

§6. Ordered Transformations in a Chain. 366

§7. The Ordinary Real and Complex Projective Spaces. 370

§8. Categorical Systems. Quadratic Irrationalities. 374

§9. List of Assumptions and their Mutual Independence. 376

Introduction.

This paper contains a completely independent[†] set of assumptions for projective geometry stated in terms of undefined elements called points and undefined classes of points called lines. The assumptions are so arranged that a certain group of eight characterize what may be called *general projective spaces*, *i. e.*, spaces in which the points can be represented by homogeneous coordinates which are elements of a finite or infinite number-system, in which the operation of multiplication may or may not be commutative. On adding to this group an assumption (like our Assumption P, p. 352) from which can be proved the fundamental theorem of projectivity (in the form given, for example, on p. 352), we obtain a set of assumptions which characterize the most general projective spaces properly so-called, *i. e.*, spaces in which the points may be represented by homogeneous coordinates which are elements of a commutative number-system, *i. e.*, of a finite or infinite field.

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[*] Presented to the American Mathematical Society, Dec. 27, 1907.

[†] Ordinally independent sets have been given before, but so far as the authors are aware this is the first completely independent set.

Modular and non-modular spaces, *i. e.*, spaces in which the coordinates are elements of modular or non-modular fields, are distinguished by means of Assumption H (§ 4). Finally it is shown how by replacing Assumption P by assumptions of continuity and closure we may arrive at categorical and completely independent sets of assumptions on the one hand for the projective space in which the coordinates are ordinary real, and on the other hand for that in which the coordinates are ordinary complex numbers.

A complete list of the assumptions for the ordinary real and complex projective spaces of three dimensions will be found at the beginning of § 9.

The obligations of the authors to previous work will be evident to any one who is familiar with the literature of the subject. For this reason we have omitted detailed references to previous work and content ourselves with the reference to the article of ENRIQUES, Prinzipien der Geometrie, in the *Encyklopaedie der Mathematischen Wissenschaften*, Band III, Part I, pp. 1–129, for a bibliography. For a similar reason we have omitted all proofs of the early theorems, believing that their derivation from the assumptions in question is sufficiently familiar. The definitions of many well-established terms have likewise been omitted in the interest of brevity.

§ 1. *The Assumptions for General Projective Geometry.*

In the following assumptions for projective geometry we have chosen the *point* and the *line* as undefined elements, the line being regarded as an undefined *class of points*. The only undefined relation used is that of *belonging to a class*. This relation will be variously expressed by such phrases as: a point is on a line; a line joins two points; three points are collinear; etc. In this section we give a set of assumptions that define what may be called *general projective spaces*, in which the points may be represented by homogeneous coordinates which are elements of a finite or infinite number-system, in which multiplication may or may not be commutative.

THE ASSUMPTIONS OF ALIGNMENT, A:

A1. *If A and B are distinct points there is at least one line containing both A and B.*

A2. *If A and B are distinct points there is not more than one line containing both A and B.*

A3. *If A, B, C are points not belonging to the same line, and if a line l contains a point D of a line joining B and C and a point E, distinct from D, of a line joining C and A, then the line l contains a point F of a line joining A and B.* (Fig. 1.)

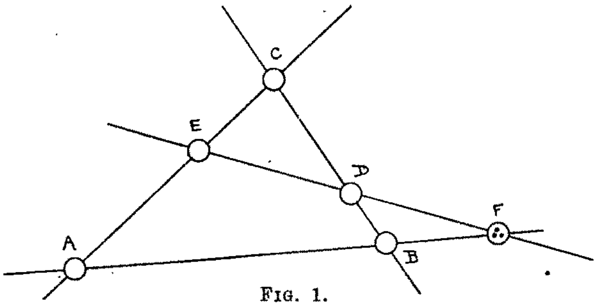

FIG. 1.

AN ASSUMPTION OF EXTENSION, E:

E0. *There are at least three points on every line.*[*]

From A1 and A2 follows readily:

THEOREM 1. *Two distinct points determine one and only one line. If C and D are distinct points of the line AB,*[†] *then A and B are points of the line CD. Two distinct lines can not have more than one point in common.*

Definition. If P, Q, R are three points not on the same line, and l is a line joining Q and R, the class S_2 of all points such that every point of S_2 is collinear with P and some point of l is called the *plane* determined by P and l. If P, Q, R, T are four points not in the same line or plane, and if α is a plane containing Q, R and T, the class S_3 of all points such that every point of S_3 is collinear with P and some point of α is called the *three-space* determined by P and α.

It is now possible to derive readily from the set of assumptions given above the results contained in the following theorems:

THEOREM 2. *If A and B are points of a plane, every point of the line AB is a point of the plane. Any two lines lying in the same plane have a point in common.*

---

[*] This excludes merely the case of a space in which every line consists of only two points.

[†] The symbol AB implies $A \neq B$ and denotes the line determined by A and B.

The plane α determined by a point P and a line l is identical with the plane determined by a point Q and a line m, if Q and m are on α. There is one and only one plane containing three given non-collinear points.

THEOREM 3. *If A and B are distinct points of a three-space, every point of the line AB is a point of the three-space. If a plane α and a line l not on α lie wholly in the same three-space, then α and l have one and only one point in common. Any two distinct planes of a three-space have one and only one line in common.*

These three theorems are meaningless unless there exists at least one line (Theorem 1), or one plane (Theorem 2), or one three-space (Theorem 3). We could proceed to define a four-space, five-space, ... , n-space in a manner analogous to the definitions of a two-space (plane) and three-space already given. The fundamental properties of alignment of such spaces can be derived without difficulty from the assumptions stated. A set of assumptions, however, from which the properties of a space of given dimensionality are to be derived, should contain in addition to those already stated such assumptions of extension and closure as will insure the existence of the space in question and exclude spaces of higher dimensionality. In this paper we confine ourselves to three dimensions. There follow accordingly for this case the necessary

ASSUMPTIONS OF EXTENSION AND CLOSURE,[*] E:

E1. *There exists at least one line.*

E2. *It is not true that every point lies on every line.*

E3. *It is not true that every point lies on every plane.*

E3′. *If S is a three-space, every point lies in S.*

It is now a simple matter to derive the principle of duality in a three-space and in a plane, in view of the fact that the duals of the assumptions can be proved without difficulty. These two principles are stated in the following two theorems:

THEOREM 4: THE PRINCIPLE OF DUALITY FOR A THREE-SPACE. *Any proposition deducible from assumptions* A *and* E *is valid if the words "point" and "plane" are interchanged.*

THEOREM 5: THE PRINCIPLE OF DUALITY IN A PLANE. *Any proposition concerning points and lines of the same plane derived from assumptions* A *and* E, *is valid if the words point and line are interchanged.*

---

[*] The words "extension" and "closure" in this connection were suggested by N. J. LENNES.

This brief statement of the principle of duality makes necessary the use of such expressions as "a line lying on a point," "a plane lying on a point or a line," "a point containing a plane" etc., in a sense that need not be further explained here.

It is now possible to enumerate the fundamental geometric forms, and to define perspectivity and projectivity[*] in the usual manner. In what follows we omit most definitions of well-established terms. Such terms as are defined, moreover, and the theorems that are proved will be confined in general to one form; the dual definitions and theorems are everywhere implied without being explicitly stated.

Of the theorems derivable from those thus far noted we mention, first:

THEOREM 6: THE THEOREM OF DESARGUES. *The intersections of the pairs of homologous sides of two perspective triangles are collinear.*

Definition. The set of points in which the sides of a complete quadrangle meet a line is called a *quadrangular set;* it is denoted by the symbol $Q(A, B, C; D, E, F)$, which implies that A, D; B, E; C, F are the intersections of pairs of opposite sides of the quadrangle with AB and that A, B, C are the intersections with AB of three concurrent sides of the quadrangle. In case $B = E$ and $C = F$, A and D are *harmonic conjugates* with respect to B and C.

From Theorem 6 then follows:

THEOREM 7. *If all but one of the points of a quadrangular set* $Q(A, B, C; D, E, F,)$ *are given, the remaining one is uniquely determined. In particular, the harmonic conjugate of a point with respect to two others is uniquely determined.*

The following propositions concerning the projectivities of one-dimensional forms are also readily derivable from the assumptions and theorems thus far noted:

THEOREM 8. *If* A, B, C, D *are points of a line, and* A', B', C' *are points of another or the same line, we always have* $(A, B, C) \overline{\wedge} (A', B', C')$[†] *and* $(A, B, C, D) \overline{\wedge} (B, A, D, C)$. *A set of collinear points which is projective with a quadrangular set is a quadrangular set. In particular, if one of two projective sets*

---

[*] We use PONCELET'S definition of projectivity, which defines it as the resultant of a sequence of perspectivities.

[†] The notation $(A, B, \ldots .) \overline{\wedge} (A', B', \ldots .)$ denotes a projectivity in which A, A'; B, B'; .... are homologous pairs. Similarly $(A, B, \ldots .) \overset{P}{\overline{\wedge}} (A', B', \ldots .)$ denotes a perspectivity with center P in which A, A'; B, B'; .... are homologous pairs.

of four collinear points is harmonic, so also is the other. If the ranges on two pairs of a set of three concurrent lines are perspective, so also are the ranges on the third pair.

It is not possible, however, to deduce from the assumptions A and E the so-called fundamental theorem of projectivity, which we state in the following form:

THE FUNDAMENTAL THEOREM OF PROJECTIVITY. *If A, B, C, D are distinct points of a line, and A', B', C' any three distinct points of another or the same line, then for any projectivities giving* $(A, B, C, D) \barwedge (A', B', C', D')$ *and* $(A, B, C, D) \barwedge (A', B', C', D'_1)$ *we have* $D' = D'_1$.

To derive the fundamental theorem another assumption is necessary, which may take any one of several forms. One form is the following:

AN ASSUMPTION OF PROJECTIVITY, P:

P. *Two projective ranges of points on two different lines which have a self-corresponding point are perspective.*

Very little use of this assumption is made in the subsequent parts of this paper; indeed *the principal part of the paper is entirely independent of it, so that all numbered theorems are derivable without its use.* We have given it here merely in order that we might characterize by a set of simple assumptions what may be called the most general *properly projective spaces; i. e.*, those in which the fundamental theorem of projectivity is valid. Such a space is characterized by assumptions A, E and P. A space satisfying assumptions A and E, and not P, may then be called an *improperly projective space.* Cf., in this connection, Theorem 14 below, which shows that assumption P is equivalent to the commutative law of multiplication in the algebra there developed.

§ 2. *Algebra of Points and the Introduction of Analytic Methods.*

At this point it seems desirable to introduce analytic methods. The introduction of a point algebra, which is possible without the use of any further assumptions, will throw more light on the preceding results and will greatly facilitate much of the subsequent work.

Given a line l and on l three distinct (arbitrary) fixed points which for convenience and suggestiveness we denote by P_0, P_1, P_∞, we define two *one-*

valued operations[*] on pairs of points of l with reference to the *fundamental points* P_0, P_1, P_∞. The fundamental points are said to determine the *scale* (P_0, P_1, P_∞) on l.

Definition. The point P_{x+y} determined by the relation $Q(P_\infty, P_x, P_0; P_\infty, P_y, P_{x+y})$ is called the *sum* of the two points P_x and P_y (in symbols $P_x + P_y = P_{x+y}$) in the scale (P_0, P_1, P_∞) on l. (Cf. Fig. 2.) The point P_{xy} determined by the relation $Q(P_0, P_1, P_x; P_\infty, P_{xy}, P_y)$ is called the *product* of P_x by P_y (in symbols $P_x . P_y = P_{xy}$) in the scale (P_0, P_1, P_∞) on l. (Cf. Fig. 3.)

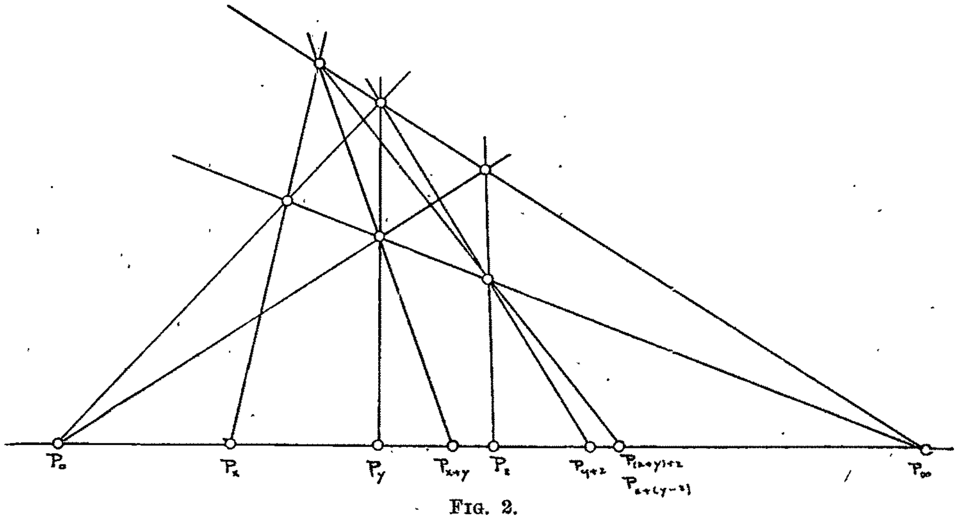

FIG. 2.

From Theorem 7 follows:

THEOREM 9. *The operations of addition and multiplication are one-valued, except for $P_0 . P_\infty$ and $P_\infty . P_0$.*

From Theorem 7 likewise follows:

THEOREM 10. *The operation of addition is commutative.*

There is no difficulty, moreover, in proving

THEOREM 11. *The operations of addition and multiplication are associative.*

For, the constructions for $(P_x + P_y) + P_z$ and $P_x + (P_y + P_z)$ can easily be so made that they are both defined by the intersection of the same line with l. Similarly for $P_x . (P_y . P_z)$ and $(P_x . P_y) . P_z$. (Cf. Figs. 2, 3.)

---

[*] By a *one-valued* operation o on a pair of points A, B is meant any process whereby with every pair A, B is associated a point C, which is unique provided the order of A, B is given; in symbols, $AoB = C$. Here "order" has no geometrical significance, but implies merely the formal difference of AoB and BoA. If $AoB = BoA$ the operation is *commutative*; if $(AoB)oC = Ao(BoC)$, *associative*.

By means of assumptions A and E alone we may also derive the following important theorem:

THEOREM 12. *Between the points P_x, P_y, P_{xy} we always have the projectivities*

$$(P_\infty, P_0, P_1, P_x) \overline{\wedge} (P_\infty, P_0, P_y, P_{xy})$$

and

$$(P_\infty, P_0, P_1, P_y) \overline{\wedge} (P_\infty, P_0, P_x, P_{xy}).$$

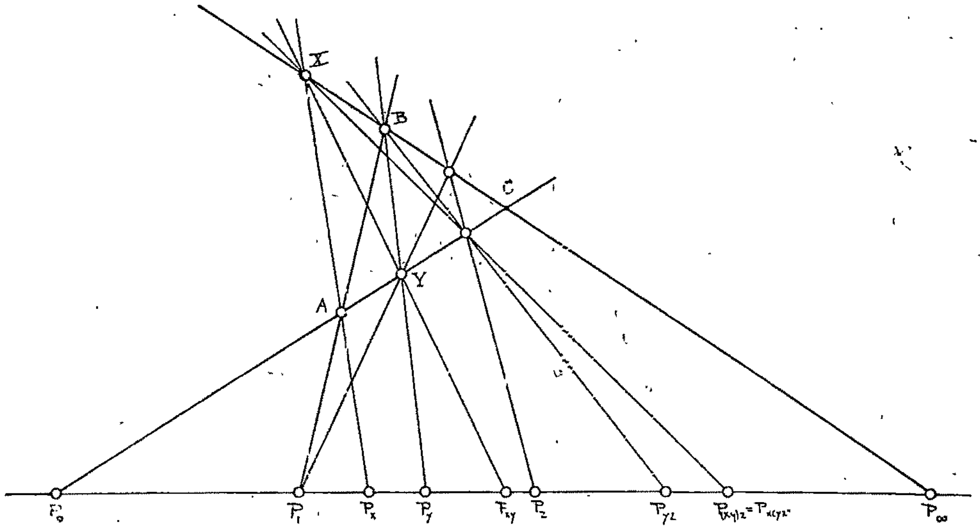

FIG. 3.

Proof. Let the quadrangle $ABXY$ define the point P_{xy}. (Fig. 3.) We then have

$$(P_\infty, P_0, P_1, P_x) \overset{A}{\overline{\wedge}} (P_\infty, C, B, X) \overset{Y}{\overline{\wedge}} (P_\infty, P_0, P_y, P_{xy});$$

and also

$$(P_\infty, P_0, P_1, P_y) \overset{B}{\overline{\wedge}} (C, P_0, A, Y) \overset{X}{\overline{\wedge}} (P_\infty, P_0, P_x, P_{xy}).$$

From this theorem we can readily derive

THEOREM 13: THE DISTRIBUTIVE LAW. *For any three points P_x, P_y, P_z on l we have $P_x \,.\, (P_y + P_z) = P_x \,.\, P_y + P_x \,.\, P_z$ and $(P_y + P_z) \,.\, P_x = P_y \,.\, P_x + P_z \,.\, P_x$.*

Proof. By Theorem 12 we have

$$(P_\infty, P_0, P_1, P_y, P_z, P_{y+z}) \overline{\wedge} (P_\infty, P_0, P_x, P_{xy}, P_{xz}, P_{x(y+z)});$$

also $Q(P_\infty, P_y, P_0;\ P_\infty, P_z, P_{y+z})$; whence we have $Q(P_\infty, P_{xy}, P_0;\ P_\infty, P_{xz}, P_{x(y+z)})$ (Theorem 8), which gives $P_{xy} + P_{xz} = P_{x(y-z)}$. This is the first relation of the theorem. The second is obtained similarly.

The commutative law of multiplication can not be derived from assumptions A and E alone. The intimate connection between the commutative law of multiplication and the fundamental theorem of projective geometry is expressed in the following:

THEOREM 14. *Multiplication is commutative, if and only if the fundamental theorem of projective geometry is valid.*

Proof. From Theorem 12 we have

$$(P_\infty, P_0, P_1, P_x) \barwedge (P_\infty, P_0, P_y, P_{xy})$$

and

$$(P_\infty, P_0, P_1, P_x) \barwedge (P_\infty, P_0, P_y, P_{yx}),$$

whence clearly $P_{xy} = P_{yx}$, if and only if the fundamental theorem holds. (Cf. p. 352.)

The inverse operations, subtraction and division, may now be defined in the usual manner. It is then readily seen that *the points of a line on which a scale has been established form a number-system,* if the point P_∞ be excluded, in which the points P_0 and P_1 play the rôle of zero and unity respectively.* For the definitions of addition and multiplication give at once

$$P_0 + P_x = P_x + P_0 = P_x, \quad P_0 \cdot P_x = P_x \cdot P_0 = P_0,$$

and

$$P_1 \cdot P_x = P_x \cdot P_1 = P_x, \text{ if } P_x \neq P_\infty.$$

This number-system is commutative, if and only if the space considered is properly projective. For convenience we shall denote the points of a line by the small letters of the alphabet, whenever we think of them as numbers of a number-system.

We may now treat analytically the projectivities on a line *for the case in which the number-system is commutative, i. e.,* for a properly projective space. It is readily seen from the definitions that each of the transformations

$$x' = x + a, \quad x' = ax, \quad x' = 1/x \tag{1}$$

defines a projectivity; and it is readily shown that every transformation of the form

$$x' = \frac{ax+b}{cx+d} \qquad (ad - bc \neq 0) \tag{2}$$

can be resolved into the product of transformations of types (1), so that every transformation (2) is a projectivity. That every projectivity in a properly projective space can be represented by (2) then follows at once from the fact that any such projectivity is uniquely defined when three pairs of homologous points are given. This leads to three linear homogeneous equations for the determination of the ratios $a:b:c:d$, and these equations are necessarily solvable in the field.

---

*For the general definition of a number-system see DICKSON, Definition of Linear Associative Algebra by Independent Postulates, *Trans. Amer. Math. Soc.*, Vol. IV (1903), p. 21.

The double ratios of four points are now defined in the usual manner and their invariance under projective transformations follows immediately from their evident invariance under each of the three types (1). Further, the double ratio of a harmonic form (a, b, c, d) in which a, c are conjugate is clearly

$$\frac{a-b}{c-b} : \frac{a-d}{c-d} = -1, \tag{3}$$

since -1 is the harmonic conjugate of 1 with respect to 0 and ∞ (by definition of -1 as $0-1$) and all harmonic forms are projective.

The exceptional character of the point P_∞ in the point-algebra may be removed in the usual manner by the introduction of homogeneous coordinates and the ordinary analytic methods may be developed for the plane and for space without difficulty.

§ 3. *Nets of Rationality.*

Definition. A point P of a line is said to be *harmonically* (*quadrangularly*) *related to three given distinct points* A, B, C of the line, provided P is one of a sequence of points A, B, C, H_1, H_2, H_3, ... of the line, finite in number, such that H_1 is the harmonic conjugate of one of the points A, B, C with respect to the other two, and such that every other point H_i is harmonic with three of (is one of a quadrangular set of which the other five belong to) the set A, B, C, H_1, H_2, ..., H_{i-1}. The class of all points harmonically related to three distinct points A, B, C on a line is called the *net of rationality* (on the line) defined by A, B, C; it is denoted by $R(A, B, C)$.

THEOREM 15. *If* A, B, C, D *and* A', B', C', D' *are respectively points of two lines such that* $(A, B, C, D) \barwedge (A', B', C', D')$, *and if* D *is harmonically* (*quadrangularly*) *related to* A, B, C, *then* D' *is harmonically* (*quadrangularly*) *related to* A', B', C'.

This follows directly from the fact that the projectivity of the theorem makes the set of points H_j which defines D as harmonically (quadrangularly) related to A, B, C projective with a set of points H_i' such that every harmonic (quadrangular) set of points of the sequence $A, B, C, H_1, H_2, \ldots, D$ is homologous with a harmonic (quadrangular) set of the sequence $A', B', C', H_1', H_2', \ldots, D'$ (Theorem 8).

COROLLARY. *If a class of points on a line is projective with a net of rationality on a line, it is itself a net of rationality.*

THEOREM 16. *If K, L, M are three distinct points of $R(A, B, C)$, A, B, C are points of $R(K, L, M)$.*

Proof. From the projectivity $(A, B, C, K) \barwedge (B, A, K, C)$ follows by Theorem 15 that C is a point of $R(A, B, K)$, or $R(A, B, C) = R(A, B, K) = R(A, K, M) = R(K, L, M)$.

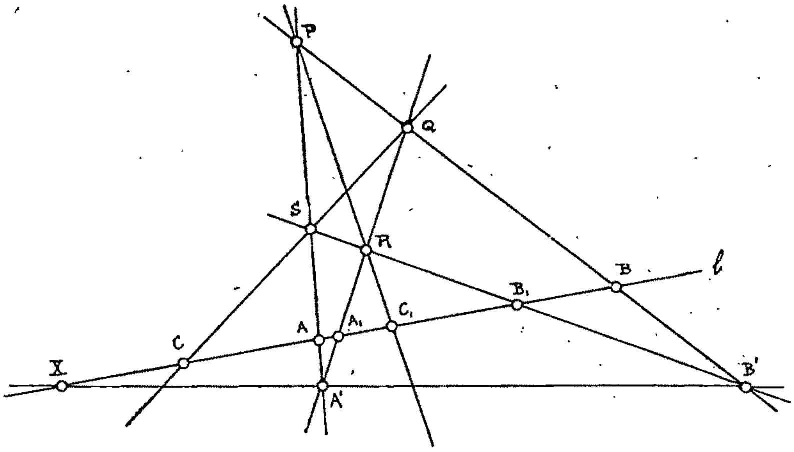

FIG. 4.

COROLLARY. *A net of rationality on a line is determined by any distinct three of its points.*

THEOREM 17. *If all but one of the six (or five, or four) points of a quadrangular set are points of the same net of rationality R, this one point is also a point of R.*

Proof. Let the sides of the quadrangle $PQRS$ (Fig. 4) meet the line l as indicated in the points A, A_1; B, B_1; C, C_1; and suppose that the first five of these are points of a net of rationality $R = R(A, A_1, B_1) = R(A_1, B_1, C) = \ldots.$ We must prove that C_1 is a point of R. Let the pairs of lines PS, QR and PQ, RS meet in A', B' respectively, and let $A'B'$ meet l in X. We then have

$$(A_1, B, B_1, C) \overset{Q}{\underset{\wedge}{=}} (R, B', B_1, S) \overset{A'}{\underset{\wedge}{=}} (A_1, X, B_1, A),$$

whence $(A_1, B, B_1, C) \barwedge (A_1, X, B_1, A)$, so that if B is a point of $R(A_1, B_1, C)$, X is a point of $R(A_1, B_1, A)$; but these two nets are identical with R, so that X is a point of R. Now,

$$(A, B_1, X, A_1) \overset{A'}{\doublebarwedge} (S, B_1, B', R) \overset{P}{\doublebarwedge} (A, B_1, B, C_1),$$

which shows that C_1 is a point of R.

Corollary. *The class of all points quadrangularly related to three distinct points A, B, C is $R(A, B, C)$.*

Although the fundamental theorem of projective geometry can not be deduced in general from the assumptions A and E, the corresponding theorem for a net of rationality on a line follows almost immediately from the preceding theorems. It may be stated as follows:

Theorem 18: The Fundamental Theorem of Projectivity for a Net of Rationality on a Line. *If A, B, C, D are distinct points of a net of rationality R on a line, and A', B', C', any three distinct points on another or the same line, then for any projectivities giving $(A, B, C, D) \barwedge (A', B', C', D')$ and $(A, B, C, D) \barwedge (A', B', C', D_1')$ we have $D' = D_1'$.*

Proof. Let Π and Π_1 be the two projectivities respectively. Then clearly the projectivity $\Pi_1 \Pi^{-1}$ leaves A', B', C' unchanged and transforms D' into D_1'. But it is easy to see that a projectivity which leaves three distinct points of a line unchanged leaves all the points of the net of rationality defined by these points unchanged, since if three points of a line are fixed the harmonic conjugate of one with respect to the other two is also fixed.

Corollary. *If two nets of rationality on different lines are projective and have a self-corresponding point, they are perspective.*

Definition. If A, B, C, D are the vertices of a quadrangle, a point P of their plane is said to be *rationally related* to them, if P is one of a sequence of points $A, B, C, D, D_1, D_2, \ldots$ finite in number, such that D_1 is a diagonal point of the original quadrangle and such that every other point D_i is a diagonal point of a quadrangle whose vertices are contained in the set

$$A, B, C, D, D_1, D_2, \ldots, D_{i-1}.$$

A line is said to be rationally related to A, B, C, D, if it joins two points rationally related to them. The class of all points and lines rationally related to four distinct coplanar points is called the *net of rationality* (in the plane) defined by the four points. It is denoted by $R(A, B, C, D)$.

The following is a consequence of this definition and the corollary of Theorem 17:

THEOREM 19. *The points in which the lines of a net of rationality in a plane meet any line of the plane form a net of rationality on this line.*

Definition. If A, B, C, D, E are the vertices of a complete space five-point, a point P is said to be *rationally related* to them, if P is one of a sequence of points $A, B, C, D, E, I_1, I_2, I_3, \ldots$, finite in number, such that I_1 is the intersection of three distinct faces of $ABCDE$, and such that every other point I_i is the intersection of three distinct faces of a complete space five-point whose vertices belong to the set $A, B, C, D, E, I_1, I_2, \ldots, I_{i-1}$. A line is said to be rationally related to A, B, C, D, E if it joins two, a plane if it joins three non-collinear, points which are rationally related to A, B, C, D, E. The set of all points, lines, and planes rationally related to A, B, C, D, E is called the *net of rationality* (in space) defined by A, B, C, D, E; it is denoted by $R(A, B, C, D, E)$.

This definition gives

THEOREM 20. *The points and lines (points) in which the lines and planes (planes) of a net of rationality in space meet any plane (line) form a net of rationality on this plane (line).*

Theorems analogous to Theorems 15, 16, 18 can readily be derived for nets of rationality in a plane and in space.

This leads to the important result:

THEOREM 21. *A net of rationality in space is a space satisfying the assumptions* A *and* E *and also* P; *i. e., a net of rationality in space is a properly projective space.*

COROLLARY. *If P_0, P_1, P_∞ are three distinct points of any line, the points of $R(P_0, P_1, P_\infty)$ form a commutative number-system or field.*

This follows directly from the last theorem in connection with Theorem 14.

"Rational" geometries would result, if we added to our assumptions A and E another assumption of closure (E3′(r)) to the effect that all the points of space belong to the same net of rationality. In general, any five-point in any properly or improperly projective space determines a sub-space which is rational and therefore properly projective.

§ 4. *Assumption* H *and the Definition of Separation.*

Definition. Any sequence of points $\ldots, H_0, H_1, H_2, H_3, \ldots$ on a line is called a *harmonic sequence*, if it has the properties: 1) that the middle one of any three consecutive points of the sequence is the harmonic conjugate, with respect to the other two, of a fixed point H of the line; and 2) that, if H_i, H_{i+1} are any two consecutive points of the sequence, the harmonic conjugate of H_i with respect to H_{i+1} and H is a point of the sequence. The point H is called the *limit point* of the sequence. (Cf. Fig. 5.)

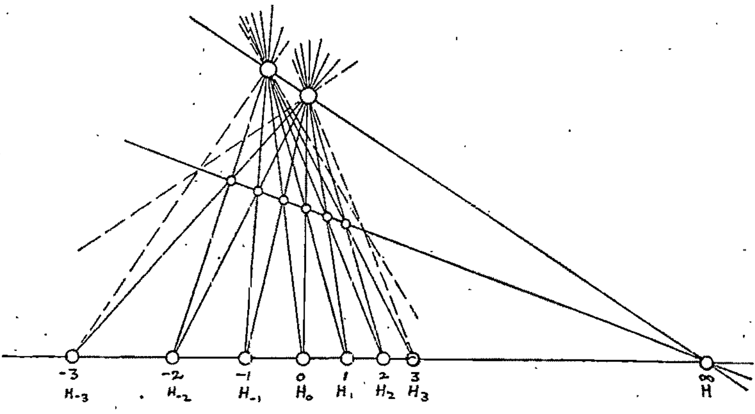

FIG. 5.

If the limit point of a harmonic sequence is associated with ∞[*] and two successive points of the sequence with 0 and 1 respectively, it follows (cf. Fig. 5) at once, from the definitions of § 2, that the sequence consists of the points

$$\dots, -1-1-1, -1-1, -1, 0, 1, 1+1, 1+1+1, \dots;$$

or if we adopt the usual symbols to denote these numbers, of the points

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots.$$

It should here be noted that the assumptions thus far made do not imply that this sequence contains an infinite number of points.

Clearly all points of a harmonic sequence belong to the same net of rationality. Moreover, it follows from Theorem 21, corollary, that if x and y belong to the net $R(0, 1, \infty)$ so also do $x+y$, $x-y$, xy, and x/y, so that $R(0, 1, \infty)$ contains all numbers that can be obtained from 0, 1 by a finite number of the rational operations. Further, from (3)[†] (p. 356) of § 2, it follows that the fourth harmonic of any point in $R(0, 1, \infty)$ with respect to two others can be obtained by a finite number of rational operations on a, b, c. Whence follows that *the number-system associated with every net of rationality consists of all numbers that can be obtained by a finite number of rational operations on* 0 *and* 1, *and only these.*

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[*] For convenience we use the symbols 0, 1, ∞, ..., x, y, ... in place of P_0, P_1, P_∞, ..., P_x, P_y, ...

[†] (3) is clearly applicable, since multiplication is commutative in any net of rationality.

Returning to the harmonic sequence, two possibilities present themselves: Either all the points of a harmonic sequence are distinct from their predecessors, in which case the number-system contains all the ordinary rational numbers; or some point of the sequence coincides with one of the preceding points, in which case the number-system consists of all integers mod. p (p being any prime).[*] The net of rationality may in this case be called *modular.* These results we combine as follows:

THEOREM 22. *Every net of rationality determines a number-system which consists either of all integers mod.* p *(*p *any prime), or of the set of all rational numbers. In the first case the whole space in which the net lies may contain either a finite or an infinite number of points, but it has the same modulus for all of its nets of rationality. In the second case the whole space and all of its nets of rationality are infinite.*

COROLLARY. *Any (not necessarily commutative) number-system is such that any two numbers* a, b, *(e. g.,* 0, 1*) determine a set of numbers rationally related to them which is either finite and prime or infinite and isomorphic with the set of all rationals.*

Working now toward the characterization of the ordinary real and complex projective spaces, we eliminate the possibility of a modular number-system by the following:

ASSUMPTION H:

H. *If there is any harmonic sequence, there is one such that every point of it is distinct from all the points of the sequence that precede it.*[†]

By virtue of this assumption we have clearly:

THEOREM 23. *The points of any net of rationality on a line give rise to a number-system which is simply isomorphic with the field of all rational numbers.*

We proceed to define a fundamental relation between pairs of points of a net of rationality on a line for which H is satisfied:

Definition. Two points A, C of a non-modular net of rationality on a line are said to *separate* two others B, D of the net (in symbols $AC \parallel BD$), if and only if the assignment of the numbers 0, 1, ∞ to the points A, B, C respectively assigns a negative number to D.

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[*] The modulus must be a prime number, since division must be always possible.

[†] This has as part of its content "Fano's Axiom," that the diagonal points of a complete quadrangle are non-collinear. Cf. GINO FANO, *Gior. di Mat.*, Vol. XXX (1892), p. 106.

This definition is dependent on the order in which the points are taken. The following theorem shows, however, that the relation of separation is independent of the order of the pairs of points or of the order of the points within the pair:

THEOREM 24. 1) *The relation* $AC \parallel BD$ *implies the relations* $BD \parallel AC$ *and* $AC \parallel DB$, *and excludes the relation* $AB \parallel CD$. 2) *Given any four distinct points of a net of rationality on a line, we have either* $AB \parallel CD$, *or* $AC \parallel BD$, *or* $AD \parallel BC$. 3) *From the relations* $AC \parallel BD$ *and* $AD \parallel CE$ *follows the relation* $AD \parallel BE$.*

This theorem follows at once from the following two:

THEOREM 25. *If* $AC \parallel BD$ *and* $(A, B, C, D) \barwedge (A', B', C', D')$, *then also* $A'C' \parallel B'D'$.

Proof. Since any projectivity transforms every quadrangular set into a quadrangular set, it is clear that the number assigned to D' by the assignment of 0, 1, ∞ to A', B', C' must be precisely the same as the number assigned to D by the assignment of 0, 1, ∞ to A, B, C.

THEOREM 26. *Two points* a, c *of the net* $R(0, 1, \infty)$ *separate two others* b, d *of this net if and only if one and only one of the numbers* a, c *lies between the two numbers* b, d.

Proof. If we project a, b, c, supposed finite, into 0, 1, ∞ respectively by the transformation

$$x' = \frac{b-c}{b-a} \cdot \frac{x-a}{x-c},$$

it is readily seen that x' is negative if and only if one of the numbers a, c, and only one, lies between b and x. The necessary modification of this argument in case one of the numbers a, b, c, d is ∞ is obvious.

COROLLARY. *Two harmonic pairs always separate each other.*

§ 5. *The Assumption of Continuity and the Definition of a Chain.*

Definition. Given three distinct points A, B, C of a net of rationality on a line, the *segment* ABC (seg ABC) of the net consists of B and all points X of

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* The properties expressed in this theorem are sufficient to define abstractly the relation of separation. Cf. VAILATI, *Rëvue de Mathématiques*, Vol. V, pp. 76, 183; also, PADOA, *Rëvue de Mathématiques*, Vol. V, p. 185.

the net such that A, C do not separate B, X. The totality of points Y such that A, C do separate B, Y constitutes the *segment complementary to* seg ABC. The points A, C are called the extremities of each of the two segments.

Clearly seg ABC and seg CBA contain the same points.

Any two distinct points of a net of rationality on a line divide the net into two segments S and S' such that the two given points separate every pair of points of which one belongs to S and the other to S', and such that no pair of points of S separates any pair of points of S'. It is clear also that any point P of a segment S (of a net of rationality on a line) of which A and C are extremities divides the segment S into two segments S_1, S_2 such that no pair of points of S_1 separates any pair of points of S_2, and such that the pair AP and the pair PC each separates every pair of points of S, of which one belongs to S_1 and the other to S_2.

Definition. Any division of the points of a non-modular net of rationality on a line into two classes K_1 and K_2 such that

1) Every point of the net belongs either to K_1 or to K_2,

2) No pair of points of K_1 separates any pair of K_2,

is called a *cut* in the net. The classes K_1, K_2 are called the *sides* of the cut.

Any two distinct points of a net of rationality on a line determine a cut, therefore, in which the two segments defined by the two points are the classes K_1 and K_2 respectively, provided the extremities of the segments be assigned to the classes K_1, K_2 in any one of the possible four ways.

From Theorem 25 follows at once that the projective transform of a cut is again a cut.

Definition. Given a cut K_1, K_2 in a net of rationality on a line, and let A_1, A_2 be any two points of K_1, K_2 respectively; then a point X of the net which divides seg A_1XA_2 into two segments S_1 and S_2 such that S_1 contains only points of K_1, and S_2 only points of K_2, is called a *cut-point* of the cut.

It is evident from this definition that a cut can not have more than two cut-points.

Definition. A cut in a net of rationality on a line is said to be *closed,* if it has two cut-points in the net; it is said to be *singly open,* if it has a single cut-point in the net; and *doubly open,* if it has none.

Any closed cut K_1, K_2 with cut-points C_1, C_2 we will denote by $K(C_1, C_2)$. Such a cut clearly divides the net of rationality into two segments S_1, S_2 such that all points of S_1 are in K_1 and all points of S_2 in K_2. A singly open cut with cut-point C we will denote similarly by $K(C)$.

We now introduce continuity into the nets of rationality on a line by the following

ASSUMPTION OF CONTINUITY, C:

> C. *If there exists any non-modular net of rationality, at least one point Q of some line l and at least one net of rationality R on l containing Q is such that associated with every singly open cut $K(Q)$ in R is a point X_k such that:* 1) *X_k is on l;* 2) *if two cuts $K_1(Q)$ and $K_2(Q)$ are distinct, the points X_{k_1} and X_{k_2} are distinct;* 3) *if two cuts $K_1(Q)$ and $K_2(Q)$ are projective, the points X_{k_1} and X_{k_2} form a homologous pair.*

THEOREM C. *The point X_k is not a point of R.*

Proof. 1) X_k is not identical with Q, by C, 2) and C, 3). 2) Suppose X_k not identical with Q but in R, and let I be the involution* with double points Q and X_k. Then $K(Q)$ is transformed into a different cut $K'(Q)$. For if A, B are points on opposite sides of the cut $K(Q)$ and in the same seg (QAX_k), they are transformed into points of the complementary segment which are evidently on the same side of $K(Q)$. Hence, by C, 3), we should have two distinct cuts having the same X_k, which is contrary to C, 2).

The last assumption then implies the existence on some one line of more than one net of rationality, and hence by projection implies the existence of more than one net of rationality on every line. It is then in contradiction with E3′(r) (cf. end of § 3), which we mentioned as an assumption of closure for "rational" spaces.

We proceed to prove the properties expressed in Assumption C and Theorem C for every net of rationality on every line. We note first that every singly open cut in any net of rationality R' on l containing Q has associated with it a unique point. For, let $K'(Q)$ be such a cut and let Π be any projectivity on l which transforms R' into R and Q into itself. The cut $K'(Q)$ is then transformed into a $K(Q)$. By this projectivity a definite point X' of l is transformed into the point X associated with this $K(Q)$. Moreover the point X' is unique; for, if Π_1 is another projectivity transforming R' into R and $K'(Q)$ into $K_1(Q)$, then $\Pi_1 \Pi^{-1}$ is a projectivity transforming $K(Q)$ into $K_1(Q)$. The supposition that X' is not unique then leads at once to a contradiction of c, 3). *We define X' to be the point associated with $K'(Q)$.* Clearly also, with this definition, we see that if any two singly open cuts $K_1(Q)$ and $K_2(Q)$ on l are distinct, the points associated with them are distinct; and that in any projectivity on l leaving Q fixed whereby two singly open cuts are projective, the associated points are homologous.

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*An involution is defined as any projectivity on a line of period two. By "the" involution mentioned is meant the one in which the transform of any point P of the line is the harmonic conjugate of P with respect to Q and X_k. This form of statement does not assume the fundamental theorem.

Given now any singly open cut $K(Q')$ in any net of rationality on any line l', let $K(Q')$ be projected into a cut $K(Q)$ on l; the point X associated with $K(Q)$ is then the transform of a definite point X' on l' which is unique by reasoning similar to that employed in the preceding paragraph. *We define X' to be the point associated with $K(Q')$.* The properties expressed by c, 2) and c, 3) are then readily seen to hold on every line in space. The point thus associated with a singly open cut we will call the *irrational cut-point* of the cut; the other cut-point is then called *rational.* The results of the preceding paragraphs are summarized in the following:

THEOREM 27. 1) *Every singly open cut in any net of rationality on any line defines a unique irrational cut-point on the line not in the net.* 2) *If two such cuts on the same line with the same rational cut-point are distinct, the irrational cut-points are distinct.* 3) *If two singly open cuts are projective, their cut-points are homologous.*

Definition. The totality of points of a net of rationality $R(A, B, C)$, together with all the irrational cut-points defined by singly open cuts $K(C)$ in $R(A, B, C)$, is called the *chain* defined by A, B, C, and is denoted by $\mathfrak{C}(A, B, C)$. The irrational cut-points are said to be *irrational* with respect to A, B, C.

From 3) of the last theorem then follows directly:

COROLLARY. *The projective transform of any chain is a chain.*

THEOREM 28. *If P, Q, R are points of the chain defined by A, B, C, then A, B, C are points of the chain defined by P, Q, R.*

Proof. As in the proof of Theorem 16 we need only show that if P is a point of $\mathfrak{C}(A, B, C)$, then C is a point of $\mathfrak{C}(A, B, P)$ and this only when P is irrational with respect to A, B, C. Let P be defined by the singly open cut $K(C)$. This cut is transformed by the projectivity $(A, B, C, P) \barwedge (B, A, P, C)$ into a singly open cut $K(P)$ of the net $R(B, A, P)$, whose irrational cut-point must (by Theorem 27, 3)) be C.

COROLLARY 1. *A chain is determined by any distinct three of its points.*

COROLLARY 2. *A chain contains the irrational cut-point of every singly open cut in any net of rationality in the chain.*

COROLLARY 3. *Every point of* $\mathfrak{C}(A, B, C)$ *irrational with respect to* A, B, C *can be defined by a singly open cut* $K(P)$, *where* P *is any point of* $R(A, B, C)$.

We can now easily derive

THEOREM 29: THE FUNDAMENTAL THEOREM OF PROJECTIVITY FOR A CHAIN. *If* A, B, C, D *are distinct points of a chain and* A', B', C' *any three distinct points of a line, then for any projectivities giving* $(A, B, C, D) \barwedge (A', B', C', D')$ *and* $(A, B, C, D) \barwedge (A', B', C', D_1')$ *we have* $D' = D_1'$.

Proof. Let Π, Π_1 be the two projectivities mentioned in the theorem. $\Pi_1^{-1}\Pi$ then leaves every point of $\mathfrak{C}(A, B, C)$ fixed; for it leaves every point of $R(A, B, C)$ fixed, and hence by Theorem 27, 3) must leave every irrational cut-point of singly open cuts in $R(A, B, C)$ fixed. But $\Pi_1^{-1}\Pi$ is then the identical transformation as far as the points of $\mathfrak{C}(A, B, C)$ are concerned; whence $D'=D_1'$.

This theorem may also be stated as follows:

Any projective correspondence between the points of two chains is uniquely determined by three pairs of homologous points.

From this theorem follows that the points of a chain determine a commutative number-system, which by reference to Assumption C will in the next section be seen to be isomorphic with the system of ordinary real numbers.

§6. *Ordered Transformations in a Chain.*

The relation of separation between pairs of points has been defined only when the four points belong to the same net of rationality on a line. We proceed to extend the definition to any four points of the same chain.

Definition. A, B, C, D being four points of the same chain and D irrational with respect to A, B, C, the pair A, C is said to *separate* the pair B, D, if and only if A, C belong to different sides of the cut $K(B)$ of $R(A, B, C)$ defining D.

This definition is justified by Corollary 3 of Theorem 28.

This relation of separation is now defined for all the points of a chain, and is readily seen to have the fundamental properties expressed in Theorem 24. For Theorem 25 clearly holds for the more general use of the term, and this leads easily to the properties mentioned.

Definition. Given any three distinct points of a chain, the *segment* ABC of the chain (Seg ABC) consists of B and all points X of the chain such that A, C do not separate B, X; the remaining points of the chain, excluding A, C, constitute the segment of the chain complementary to Seg ABC. In the sequel the word "segment" will always mean segment of a chain, unless otherwise specified.

Clearly Seg ABC and Seg CBA contain the same points.

Any two distinct points of a chain then divide the chain into two complementary segments such that the given points separate every pair of points of the chain of which one lies in one of the segments and the other in the other segment. Conversely, whenever the points of a chain fall into two classes K_1, K_2 such that every point of the chain belongs either to K_1 or to K_2 and such that no pair of points of K_1 separates any pair of K_2, there exist two points of the chain which divide the chain into two segments S_1, S_2 such that every point of S_1 is a point of K_1 and every point of S_2 a point K_2.

We may now readily define *order* in a chain. We have seen that Seg ABC and Seg CBA contain the same points. Corresponding, however, to the two symbols $\overline{ABC}$ and $\overline{CBA}$ we distinguish two *orders* in the segment.

Definition. If two points P, Q are two points of the segment ABC of a chain, P is said to *precede* $Q\,(P < Q)$ in the order $\overline{ABC}$ if and only if $AQ \parallel PC$; Q is then said to *follow* $P\,(Q > P)$. Further, A is said to *precede* and C to *follow* every point of the segment in the order $\overline{ABC}$. The phrase "P, Q, ..., etc., are points of the directed segment $\overline{ABC}$" will in the sequel imply that P, Q, etc., are points of the segment ABC and that the statement $P < Q$ means "$P < Q$ in the order $\overline{ABC}$."

This relation of *linear order* ($<$) is at once seen to satisfy the following conditions:

THEOREM 30. 1) *If we have $P < Q$ in a given order, then $Q < P$ is impossible in that order.* 2) *If we have $P \mathrel{+\!\!\!\!=} Q$, then in a given order we have either $P < Q$ or $Q < P$.* 3) *If P, Q, R are points of a directed segment $\overline{ABC}$ such that we have $P < Q$ and $Q < R$, then we have $P < R$.**

From the definition of order it follows that if P precedes Q in the directed segment $\overline{ABC}$, then Q precedes P in the directed segment $\overline{CBA}$. The order in these two directed segments is therefore said to be *opposite.*

A chain is now seen to have the following fundamental property:

THEOREM 31. *If the points of a directed segment of a chain be divided into two classes H_1, H_2 such that every point of the segment belongs either to H_1 or to H_2,*

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*These three properties are sufficient to define linear order abstractly. Cf. HUNTINGTON, The Continuum as a Type of Order, *Annals of Mathematics*, Vol. VI (1905), p. 151.

and such that every point of H_1 precedes every point of H_2, then there exists one point M of the segment such that every element which precedes M is a point of H_1 and every point which follows M is a point of H_2.

Definition. A sequence of points $P_1, P_2, P_3, \ldots, P_n$ of a chain is said to be an *ordered sequence*, if they are points of a directed segment such that

$$P_1 < P_2 < P_3 \ldots < P_n.$$

Any three points of a chain are an ordered sequence, but any four points are not.

THEOREM 32. *If A, B, C, D are an ordered sequence, so also are B, C, D, A.*

Proof. By definition we have $AC \parallel BD$ in the directed segment $\overline{ACD}$; whence in the directed segment $\overline{BDA}$, we have $BD \parallel CA$.

COROLLARY. *If $P_1, P_2, P_3, \ldots, P_{n-1}, P_n$ form an ordered sequence, so likewise do $P_i, P_{i+1}, P_{i+2}, \ldots, P_n, P_1, P_2, P_3, \ldots, P_{i-1}$* and *$P_i, P_{i-1}, \ldots, P_2, P_1, P_n, P_{n-1}, \ldots, P_{i+1}$.*

Hence, given any ordered sequence of points of a chain and starting with any one of the points, it is possible to write them so as to form an ordered sequence in two and only two ways. This is expressed by saying that we can take the points in two different *directions*, which are opposite.*

Definition. A transformation which transforms every ordered sequence into an ordered sequence is called an *ordered transformation.* In all that follows, the word transformation denotes a correspondence which is single-valued (one-to-one) and whose inverse is also single-valued.

From Theorem 25 we have at once:

THEOREM 33. *Every projectivity on a line is an ordered transformation.*

Definition. In the number-system determined by the scale $(0, 1, \infty)$ on a chain a number a is said to be *less than* a number b, if $a < b$ in the order $\overline{01\infty}$.

THEOREM 34. *The number-system determined by the scale $(0, 1, \infty)$ in a chain is isomorphic with the system of real numbers.*

Proof. This theorem may be conveniently established by referring to a set of postulates descriptive of the real number-system. We shall use the set given by HUNTINGTON in Vol. VI (1905), p. 39, of the *Transactions of the American*

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* This establishes the so-called "cyclical order" in a chain. Cf. ENRIQUES' assumption, *Vorlesungen über projektive Geometrie*, Leipzig (1903), p. 23.

Mathematical Society. That HUNTINGTON'S *I, II, A*1–*A*6, *M*1, *M*2, *AM*1 are satisfied is equivalent to the fact that we have to do with a commutative number-system, which is a consequence of Theorem 29. In consequence of the definition above and Theorem 30 the elements of this number-system satisfy the magnitude relations "greater and less than" and the postulate of continuity. This verifies HUNTINGTON'S *R*1–*R*6. The projectivities $x' = x + a$ and $x' = ax$ transform Seg $(-a\,0\,\infty)$ into Seg $(0\,a\,\infty)$ and Seg $(0\,1\,\infty)$ into Seg $(0\,a\,\infty)$ respectively. This, in connection with Theorem 33, shows that if $a > 0$ and $b > 0$ then $a + b > 0$ and $ab > 0$. In like manner if $a < 0$ and $b < 0$ then $a + b < 0$. This verifies HUNTINGTON'S *RA*1, *RA*2, *RAM*1, and completes the list of assumptions which he uses to characterize the system of real numbers categorically.

We consider now a projective transformation of a chain into itself. Such a transformation is ordered, but the directions of the transformed sequences may or may not be the same as those of the original sequences. If the direction in a chain is preserved by a transformation, the latter will be called *directly ordered,* or simply *direct;* otherwise, if ordered, it is *oppositely ordered,* or *opposite.*

The analytical condition that a projective transformation of a chain into itself be direct or opposite is now readily obtained. Let the chain be $\mathfrak{C}(01\infty)$. We have already seen that for any class of points forming a commutative number-system any projectivity is given by

$$x' = \frac{ax + b}{cx + d}, \qquad D = ad - bc \neq 0. \tag{1}$$

If $\mathfrak{C}(01\infty)$ is transformed into itself it is clear that a, b, c, d are all real numbers. The projectivity $x' = x + b$ is direct, $x' = \frac{1}{x}$ is opposite, while $x' = ax$ is direct or opposite according as a is positive or negative. The desired condition given in the following theorem is then obtained at once by recalling the theorem that the determinant of the product of two projectivities is equal to the product of their determinants.

THEOREM 35. *A projective transformation* (1) *transforming* $\mathfrak{C}(01\infty)$ *into itself is direct or opposite according as D is positive or negative.*

Definition. A point which is made to correspond to itself by a transformation is called a *double point* of the transformation. A projectivity which transforms a chain into itself is said to be *hyperbolic, parabolic,* or *elliptic in the chain* according as it has two, one, or no double points in the chain.

The double points of a projectivity (1) transforming $\mathfrak{C}(01\infty)$ into itself, if they exist, are given by the roots of the equation $cx^2 + (d - a)x - b = 0$, where a, b, c, d are real. This equation has roots in the chain if and only if the discriminant

$$\begin{aligned}\Delta &= (d - a)^2 + 4bc \\ &= (d + a)^2 - 4D\end{aligned}$$

is positive or zero.

From this follows at once:

THEOREM 36. *In a chain,* 1) *every opposite projectivity is hyperbolic;* 2) *every parabolic or elliptic projectivity is direct.*

The proof of this theorem demands at some point a continuity argument We have chosen to borrow the desired result from the theory of functions of a real variable. It can, however, be proved directly from our assumptions without difficulty. We refer for this proof to ENRIQUES, *Vorlesungen über Projektive. Geometrie,* Leipzig (1903), pp. 72, 100.

Also from the definitions preceding we have at once:

THEOREM 37. *A hyperbolic projectivity in a chain is opposite or direct according as pairs of homologous points do or do not separate the double points.*

From the consideration of the fundamental cross-ratio it follows easily that if an involution (*i.e.*, a projectivity of period two) which transforms a chain into itself has a double point in the chain, it has another, and that the double points separate harmonically every pair of conjugate points. From the last two theorems and Theorem 26, corollary, then follows:

THEOREM 38. *An involution in a chain is direct and elliptic in the chain or opposite and hyperbolic, according as two pairs of conjugate points do or do not separate each other.*

Since an involution in a chain is determined by two pairs of conjugate points, the existence of both kinds of involutions follows.

§ 7. *The Ordinary Real and Complex Projective Spaces.*

We can now conveniently add the further assumptions necessary to characterize completely 1) the ordinary real projective space, or 2) the ordinary complex projective space, of three dimensions. Analytically this is equivalent to the identification of our number-system with 1) the system of ordinary real numbers, or 2) the system of ordinary complex numbers.

1). To characterize the ordinary real projective space we add simply the following assumption (of closure):

ASSUMPTION R. *There is not more than one chain on a line.*

A fundamental consequence of this assumption is the existence of projectivities on a line without double points. In fact any involution on the line determined by two pairs of conjugate points which separate each other is of this kind (Theorem 38).

2). On the other hand, to characterize completely the ordinary complex projective space we need only *replace* ASSUMPTION R by the following, ASSUMPTIONS I.

ASSUMPTION I1. *If there is a harmonic form, there is one* ($ABA'B'$) *such that one involution* I *having* AA' *and* BB' *as conjugate pairs has a double point on the line* AB.

By Theorem 26, corollary, and Theorem 38, the involution I has no double points in the chain $\mathfrak{C}(ABA')$; this assumption then implies the existence of more than one chain on the line AB. Assumptions R and I1 are then mutually contradictory (in connection with the assumptions already made).

ASSUMPTION I2.[*] *Through a point* P *of a chain* $\mathfrak{C}$ *on a line* l *and any point* J *of* l *not on* $\mathfrak{C}$ *there is not more than one chain that has no other point in common with* $\mathfrak{C}$ *than* P.

We are now in a position to prove that our number-system is indeed isomorphic with the ordinary system of complex numbers. We will show first that every point of the line l is given by the expression $A + JB$, where A, B are points of $\mathfrak{C}$ and J is a fixed point not on $\mathfrak{C}$.

Let the point P of $\mathfrak{C}$ be labelled ∞ and let any pair of points of $\mathfrak{C}$ distinct from P be labelled 0 and 1 respectively. The points of $\mathfrak{C}$ are then isomorphic with the system of real numbers and ∞ (Theorem 34). *Without assuming the commutativity of multiplication* it is readily seen that

$$x' = x + a, \quad x' = ax, \quad x' = xa, \quad x' = x^{-1}$$

each define a projectivity when a is constant. This follows easily from the Figs. 2, 3 and 6 (cf. Theorem 12). The totality of points $A + J$, where A stands for any point of $\mathfrak{C}$, therefore constitutes a chain $\mathfrak{C}_1$, by Theorem 27, corollary. This chain has no point in common with $\mathfrak{C}$ besides P, because if $A + J = B$, where $B (\neq \infty)$ is a point of $\mathfrak{C}$, we should have $B - A = J$, which would make J a point of $\mathfrak{C}$.

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[*] I2 is an assumption of closure.

Now let X be any point of l not in $\mathfrak{C}$ or $\mathfrak{C}_1$. The chain $\mathfrak{C}(XJP)$ has by 12 a point x_1, $\neq P$, in common with $\mathfrak{C}$. The projectivity $x' = x + J(1 - x_1^{-1}x)$, $(x_1 \neq 0)$, transforms $\mathfrak{C}$ into $\mathfrak{C}(XJP)$, so that every point of the latter and hence X is of the form $A + JB$, where A, B are points of $\mathfrak{C}$. If $x_1 = 0$, the projectivity $x' = Jx$ shows likewise that X is of the desired form $(A = 0)$. The points of $\mathfrak{C}_1$ also have this form. The desired result is then established.

We shall now prove the fundamental theorem of projectivity for all the points of our complex line by showing that the number-system determined on the line is commutative; that the latter is isomorphic with the system of ordinary complex numbers will then follow at once.

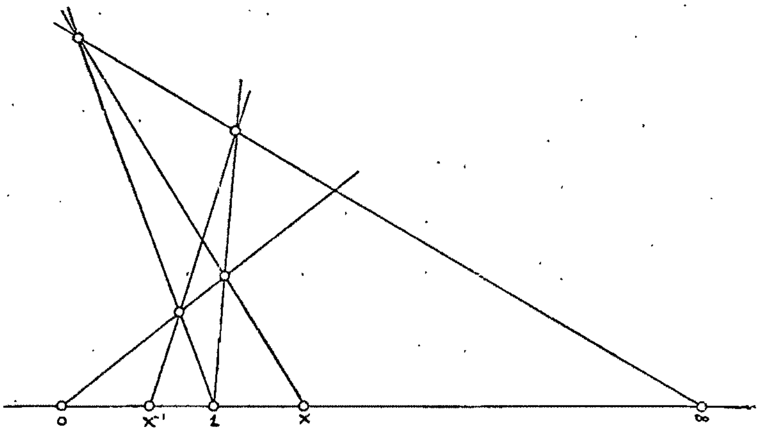

FIG. 6.

Let the points A, B, A' of 11 be labelled $0, 1, \infty$, so that the chain $\mathfrak{C}(ABA')$ is made isomorphic with the system of ordinary real numbers (and ∞), and let the double point of I in 11 be denoted by i. By the result just established all the points of the line are of the form $x + iy$, where x, y are real, since i is not on the chain $\mathfrak{C}(ABA')$. Moreover, two points $a + ib$ and $c + id$ are identical, if and only if $a = c$ and $b = d$, if a, b, c, d are real; for the equality $a + ib = c + id$ implies the relation $i = (c - a)(b - d)^{-1}$, if $b - d \neq 0$. Now, each of the projectivities $x' = ix$ and $x' = xi$, evidently transforms the chain $\mathfrak{C}(01\infty)$ into the chain $\mathfrak{C}(0i\infty)$; this gives $xi = ix_1$, where x, x_1 are real. Also each of the projectivities $x' = (1 - i)x$ and $x' = x(1 - i)$ transforms $\mathfrak{C}(01\infty)$ into $\mathfrak{C}(0, 1 - i, \infty)$, whence at once

$$x(1 - i) = (1 - i)x_2,$$

$x_1\,x_2$ being real, or by the distributive law (Theorem 13),

$$x - xi = x_2 - ix_2,$$

or by the above,

$$x - ix_1 = x_2 - ix_2,$$

or finally,

$$x = x_2 = x_1.$$

This gives $xi = ix$, for any real x, and hence follows readily the *commutativity of multiplication for any two of the numbers* $x + iy$. This in connection with Theorem 14 proves the following:

THEOREM 39. *The fundamental theorem of projectivity holds for all the points of a complex line.*

For, if it is valid on one line it is valid on every line by projection.

In view of the last theorem I is the only involution having AA' and BB' as conjugate pairs and is given by $x' = -\frac{1}{x}$; this gives at once $i^2 = -1$, and completes the proof of:

THEOREM 40. *The number-system on a line in the complex space is isomorphic with the system of all complex numbers and* ∞.

It is interesting to note here the well-known fact that whereas the property of transforming any quadrangular set into such a set is necessary and sufficient to characterize projective transformations on a line in the real geometry, it is not sufficient in the complex.

Suppose we have a transformation f which leaves the points 0, 1, ∞ fixed and transforms quadrangular sets into quadrangular sets. It is then necessarily an *ordered transformation* subject to the following functional conditions:

$$f(x+y) = f(x) + f(y),\ f(xy) = f(x)f(y),\ f(0) = 0,\ f(1) = 1,\ f(\infty) = \infty.$$

From the equation $f(x+1) = f(x) + 1$ then follows at once that $f(a) = a$, where a is any positive integer; from $f(x) + f(-x) = 0$ follows the same relation when a is any negative integer; from $f(x)\,f(1/x) = 1$ then follows readily $f(x/y) = f(x)/f(y)$, whence follows at once the relation $f(a) = a$, where a is any rational fraction or zero. From the last relation and the fact that f is ordered then follows at once the fact that *f leaves every real number fixed.* But this is sufficient to identify any transformation which transforms quadrangular sets into such sets with a projectivity on the real line. For the complex line we have at once $f(x + iy) = x + f(i)y$. Let $f(i) = a + ib$, where a and b are real, then $f(i)\,f(i) = -1$ gives $a^2 - b^2 + 1 + 2abi = 0$, whence $a = 0$, or

$b = 0$; the latter leads to the impossible relation $a^2 + 1 = 0$; the former gives $f(i) = \pm i$. By Theorem 40, $f(i) = i$ alone gives a projectivity; the relation $f(i) = -i$ leads to the so-called *anti-projectivities* of SEGRE.*

§ 8. *Categorical Systems. Quadratic Irrationalities.*

It is now very easy to see that our sets of assumptions for real and for complex projective geometry are categorical.† Confining our attention to the real case, it is clear that in any space satisfying assumptions A, E, H, C, R it is possible to establish a system of homogeneous coordinates such that every point is denoted by the ratios $x_1 : x_2 : x_3 : x_4$, where the x_i are real numbers. Therefore, given any two such spaces, a one-to-one reciprocal correspondence is set up between them in such a way as to preserve all geometrical relations, provided each point in one space corresponds to a point with the same coordinates in the other space. Since it is possible to choose the tetrahedron of reference and the point (1, 1, 1, 1) in ∞^{15} ways (corresponding to the collineations of the general projective group), we have the following:

THEOREM 41. *Any two spaces which satisfy assumptions* A, E, H, C, R *are simply isomorphic in* ∞^{15} *ways.*

In like manner is proved:

THEOREM 42. *Any two spaces which satisfy assumptions* A, E, H, C, I *are simply isomorphic in* ∞^{15} *ways.*

The following considerations will help to make clear the bearing of Assumptions C, R and I. The points and lines of a two-dimensional net of rationality form in their relations among themselves a projective plane (Theorem 21) and may be discussed either by synthetic methods or by an analytic geometry in which the coordinates are rational numbers. Corresponding lines of two projective non-perspective pencils of lines in the net intersect in a set of points in the net which lie on a conic section. This conic is said to *belong to the net.* Denote such a conic by C. Let us now recall the definitions of addition and multiplication (p. 353) which require that x and y, 0 and $x + y$ shall be pairs of an involution of which ∞ is a double point, and that x and y, 1 and xy, 0 and ∞ shall be pairs of another involution. Projecting these involutions upon the conic C we have by a familiar theorem that the lines joining corresponding points of the involutions must meet in a point, through which pass the tangents at the double points, provided double points exist (cf. Figs. 7, 8). Therefore to construct the square roots of a number z, it is necessary to construct the tangents to C from the point of intersection of the lines 0∞ and $1z$. If z is -1, then 1 and -1 harmonically divide 0 and ∞, *i. e.*, the line $0\,\infty$ passes through the point of intersection of the tangents at 1 and -1. The existence of $\sqrt{-1}$ depends, therefore, upon the possibility of drawing a tangent to a conic section from the point P of intersection of two chords of the conic, each of which passes through the polar point of the other. Assumption I1 states that this is possible. Assumption R states that it is not possible. In the geometry in which R holds P is an inside point of the conic.

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* SEGRE, Un Nuovo Campo di Ricerche Geometriche, *Torino Atti*, Vol. XXV (1890), pp. 276, 430; Vol. XXVI (1891), pp. 35, 592.

† For a discussion of this mathematico-logical term see VEBLEN, HUNTINGTON'S Types of Serial Order, *Bull. American Math. Soc.*, Vol. XII (1906), p. 303.

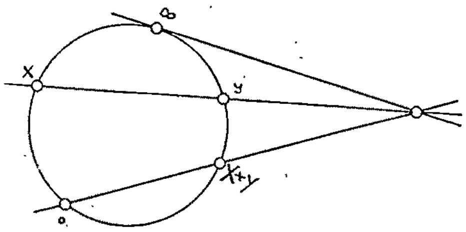

FIG. 7.

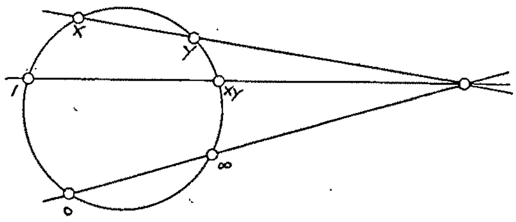

FIG. 8.

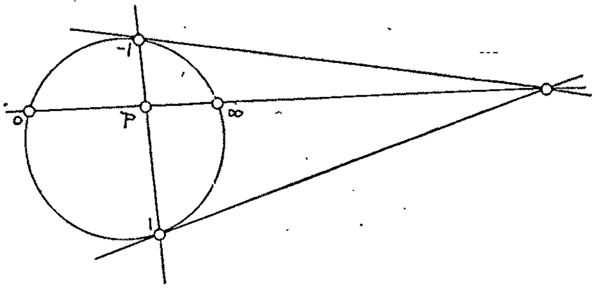

FIG. 9.

For a conic associated with a net of rationality, as C is above, the interior and exterior may be defined as follows: The order of the rational points of the conic having been determined by projection from the order of the rational points on any line, draw two lines through any point P not on the conic, the first meeting the conic in A_1, A_2, and the second meeting the conic in B_1, B_2. If A_1, A_2 separate B_1, B_2 the point P is an interior point; if not, an exterior point.

Many of the purposes of elementary projective geometry are served by operations which do not introduce into the coordinates of the points considered irrationalities of more than a certain degree. For such purposes it is not necessary to assume as much as Assumption C. The presence of the rational points is assured by Assumptions A, E and H. To adjoin to this field the operation $\sqrt{x}$, where x is positive, we may assume instead of C:

If P is any point of a two-dimensional net of rationality R exterior to a conic C belonging to the net, then there is at least one tangent to C which passes through P

Equivalent statements to this are:

An involution in which points of a given net of rationality are paired with points of the same net, and in which two conjugate pairs do not separate each other, has at least one double point. (The latter statement is readily seen to be equivalent to the former by letting the involution lie on a conic.)

A line joining two points of R, one interior to C and one exterior, meets C in at least one point.

§ 9. *List of Assumptions and their Mutual Independence.*

The following is a list of our assumptions for ordinary real projective space. The page references are to the definitions of terms occurring in the assumptions.

A1. *If A and B are distinct points, there is at least one line containing both A and B.*

A2. *If A and B are distinct points, there is not more than one line containing both A and B.*

A3. *If A, B, C are points not belonging to the same line, and if a line l contains a point D of a line joining B and C and a point E, distinct from D, of a line joining C and A, then the line l contains a point F of a line joining A and B.*

E0. *There are at least three points on every line.*

E1. *There exists at least one line.*

E2. *It is not true that every point lies on every line.*

E3. *It is not true that every point lies on every plane.* (P. 349.)

E3′. *If S is a three-space, every point lies in S.* (P. 349.)

H. *If there is a harmonic sequence, there is one such that every point of it is distinct from all the points of the sequence that precede it.* (P. 359.)

C. *If there exists any non-modular net of rationality, at least one point Q of some line l and at least one net of rationality R on l containing Q is such that associated with every singly open cut $K(Q)$ in R is a point X_k such that:* 1) *X_k is on l;* 2) *if two cuts $K_1(Q)$ and $K_2(Q)$ are distinct, the points X_{k_1} and X_{k_2} are distinct;* 3) *if two cuts $K_1(Q)$ and $K_2(Q)$ are projective, the points X_{k_1} and X_{k_2} form a homologous pair.* (Pp. 356, 363.)

R. *There is not more than one chain on a line.* (P. 365.)

For the ordinary complex projective space, Assumption R is replaced by the following two:

I1. *If there is a harmonic form, there is one $(ABA'B')$ such that one involution having AA' and BB' as conjugate pairs has a double point on the line AB.* (Pp. 351, 364, footnote.)

I2. *Through a point P of a chain $\mathfrak{C}$ on a line l and any point J of l not in $\mathfrak{C}$ there is not more than one chain that has no other point in common with $\mathfrak{C}$ than P.* (P. 365.)

We are now to prove that the assumptions above given are mutually independent, *i. e.*, such that no one of them is a formal logical consequence of the remaining ones. The method of doing this is fully explained in connection with Assumption A1, and is only sketched in the other cases.

ASSUMPTION A1. Consider the four letters A, B, C, P. Call them pseudo-points and call the set of three A, B, C a pseudo-line. The whole set A, B, C, P may be called a pseudo-space. Now, if the words "point" and "line" in the assumptions are taken to refer to these pseudo-points and line, it is evident that A1 is a false proposition, because there is no line containing both A and P. On the other hand A2 is a true proposition because there is only one line in all. A3 is true, though trivial. E0, E1, E2 are clearly true; E3 is true because no plane exists (cf. definition, p. 349). The hypotheses of Assumptions E3$'$, H, C, I$_1$ and I$_2$ are not satisfied by our pseudo-space. To introduce a technical phrase due to HUNTINGTON for the condition here met, Assumptions E3$'$, H, I, and C are "vacuously satisfied," or, as we may say more briefly, are "vacant." Clearly Assumption R is true.

Now any proposition which is a logical consequence of Assumptions A2, A3, E, H, C and R (or I) either must be true of our pseudo-space or may be vacant because involving in its deduction one or more of the vacant assumptions. The proposition A1 is neither true nor vacant of our pseudo-space, but false. Therefore A1 is not a logical consequence of the other assumptions.

ASSUMPTION A2. Let the pseudo-space consist of the points of an ordinary plane, and let all the usual lines be pseudo-lines, but in addition to these let all the points of the plane constitute a pseudo-line. In this pseudo-space every three points are collinear; hence there exists no plane. It is then readily seen that Assumptions A1, E0, E1, E2 and R are true, while A3, E3, E3′, H, C and I are vacant. Clearly also A2 is false for this pseudo-space. This proves A2 independent of all the other assumptions.

ASSUMPTION A3. Let the pseudo-points consist of the nine digits 1, 2, ... , 9; and let each row, each column, and each term in the determinant expansion of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

constitute a pseudo-line. In this pseudo-space A3 is false, as can readily be verified. A1, A2, E0, E1, E2, E3, E3′ and R are true; H, C, I1 and I2 are vacant.

ASSUMPTION E0. Let the pseudo-space consist of four pseudo-points, where the pseudo-lines are the pairs of pseudo-points. It follows that the planes are triples of pseudo-points. E0 is false, H, C and I1, I2 are vacant, while all the other assumptions are true.

ASSUMPTION E1. Let the pseudo-space consist of one pseudo-point and no pseudo-lines. All the assumptions are vacant except E1, which is false.

ASSUMPTION E2. Let the pseudo-space consist of three pseudo-points A, B, C and one pseudo-line ABC. Here E2 is false. A1, A2, E0, E1 and R are true; A3, E3, E3′, H, C, I1 and I2 are vacant.

ASSUMPTION E3. Let the pseudo-space consist of all the points of a single real (complex) projective plane, and let the pseudo-lines consist of the lines of this plane. All the assumptions for real (complex) projective geometry are true except E3, which is false, and E3′, H, C, I, which are vacant.

ASSUMPTION E3′. Let the pseudo-space be an ordinary real (complex) projective space of four dimensions. Its points may, for example, be described analytically as consisting of all sets of five homogeneous real (complex) coordinates $(x_1, x_2, x_3, x_4, x_5)$, except $(0, 0, 0, 0, 0)$, the lines being the sets of all points which satisfy three linear homogeneous equations. For such a space all the assumptions for real (complex) geometry are true, except E3′, which is false.

ASSUMPTION H. Let the pseudo-space consist of all sets of four homogeneous coordinates which are ordinary integers reduced modulo 2. In this pseudo-space H is false, C, I1 and I2 are vacant, while all the other assumptions are true.*

ASSUMPTION C. Let the pseudo-space consist of all sets of four homogeneous coordinates, except (0, 0, 0, 0), which consist of rational numbers only. Since all the assumptions for real geometry are true of this space except C, this proves the desired independence in case of the real geometry. For the complex geometry let the coordinates consist of all numbers of the form $A + Bi$, where A, B are rational. That parts 2) and 3) of C are independent of 1) and the other assumptions may be seen as follows:

C, 2). Let $[x]$ be a set of irrational numbers, such that every irrational number is of the form $ax + b$, where a, b are rational, and such that none of the numbers x is rationally related to any other x'; *i. e.*, that there is no relation of the form $x' = ax + b$, where a and b are rational.† Now, let the pseudo-space consist of a three-dimensional projective space of points, whose coordinates are rational complex numbers. Using non-homogeneous coordinates let the line l be the line $y = 0$, $z = 0$ and let the point Q be ∞. Then with the cut in the ordinary rational numbers which determines the number $ax + b$ in the ordinary geometry associate the number $a + ib$. The same number $a + ib$ is then associated with an infinitude of distinct cuts, contrary to C, 2). All the other assumptions, including C, 1) and C, 3), are satisfied except I2, which is vacant.

C, 3). Let the pseudo-space consist of the points of ordinary real or complex projective space, and let $K_1(Q)$ and $K_2(Q)$ be any two singly open cuts on l, and X_{k_1} and X_{k_2} the cut-points determined by them in the ordinary geometry. In the pseudo-space associate X_{k_1} with $K_2(Q)$ and X_{k_2} with $K_1(Q)$, and let all other irrational points be associated with their proper cuts in the ordinary way. C, 3) is then false, while the other two parts of C and all the other assumptions for the real or complex projective geometry remain true.

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* For a detailed discussion of such finite spaces, cf. VEBLEN and BUSSEY, *Trans. Am. Math. Soc.*, Vol. VII (1906), pp. 241–259.

† The assumption of the existence of a set $[x]$ is closely related to ZERMELO's assumption of the existence of an "ausgezeichnetes Element" in any class, though our assumption is weaker. It may be stated as follows: Let $R(x)$ denote the class of all numbers of the form $ax + b$, where a and b are any rational numbers, and x is a given irrational number. Any two distinct classes $R(x)$ are then mutually exclusive. Consider the class of a classes $R(x)$. Our assumption above then states that there exists a class $[x]$ of numbers which contains one and only one number from each of the classes $R(x)$, and no others. The class of classes $R(x)$ has the same cardinal (Mächtigkeit) as the continuum, whereas ZERMELO's assumption has reference to the class of all subclasses of the continuum, whose cardinal is greater than that of the continuum.

ASSUMPTION R. Let the pseudo-space consist of the three-dimensional projective space in which the coordinates are ordinary complex numbers.

ASSUMPTION I1. Let the pseudo-space consist of the three-dimensional projective space, in which the coordinates are ordinary real numbers. All the other assumptions for the complex geometry are true, except I1, which is false, and I2, which is vacant.

ASSUMPTION I2. Let the pseudo-space consist of all sets of four homogeneous coordinates, excluding (0, 0, 0, 0), which are marks of a field, F, consisting of the ordinary complex numbers together with an additional unit, t, and all algebraic functions of these.

PRINCETON UNIVERSITY.

On the Pentastroid.[*]

By R. P. Stephens.

I. *Introduction.*

In an article entitled "On a System of Parastroids," in the July number of the *Annals of Mathematics*, the equations of the curves arising from the Wallace lines were found to be of the form

$$t^n + xt^{n-1} + a_1 t^{n-2} + a_2 t^{n-3} + \ldots + b_2 t^3 + b_1 t^2 + yt + 1 = 0,$$

where x and a_i have for conjugates y and b_i respectively, and t is a parameter which is limited to the unit circle. In the particular case where $n = 3$, this is the equation of a deltoid, or hypocycloid of three cusps; and where $n = 4$, it is the equation of the parastroid. I propose to discuss the nature of the curve when $n = 5$, but I shall also call attention to a few theorems which are true for the general case. The coordinates used are circular or conjugate, however most of the work will be done by means of mapping from the unit circle.

II. *Mechanical Construction.*[†]

When $n = 5$, the equation given above takes the form

$$t^5 + xt^4 + at^3 + bt^2 + yt + 1 = 0,$$

which may be written

$$xt^3 + y = t^3(-t - a/t) + (-1/t - bt).$$

This second form is obviously the equation of a straight line which always passes through the point

$$x = -t - a/t.$$

But if t is allowed to vary, then this point traces out an ellipse. Whence we see that, if a line be fixed to the generating point of an ellipse and given the proper rotation about this point, its envelope will be the curve in question. If, in Fig. 1, the distance of P from M is μ, and if the rotation of the line l about P is 3/2 that of M about O, then the equation of l is

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[*] A preliminary report of this article was made to the American Mathematical Society, Feb. 23, 1907.

[†] Cf. *Transactions of the American Mathematical Society*, Vol. VII (1906), pp. 207–227.

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha yt - \alpha = 0, \tag{1}$$

where α is the clinant of the line l when $t = 1$. The equation of the ellipse generated at the same time is

$$x = t + \mu/t. \tag{2}$$

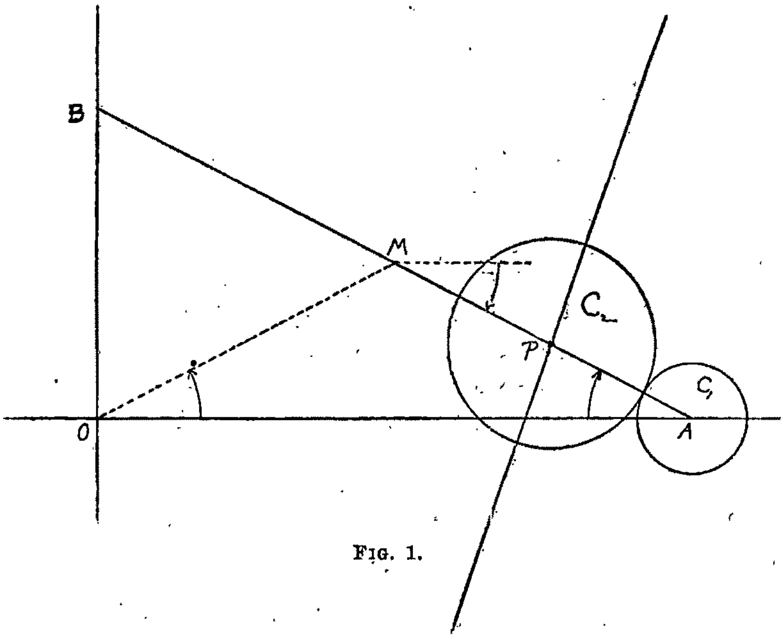

FIG. 1.

In Fig. 2 is a diagram of the instrument as used in the construction of the figures which follow. The gears G_1 and G_2 are to each other as 1 : 2. The first is centered on AB and OA and does not rotate; while G_2, centered on AB, rotates about P.

If a different combination of gears—say G_5, G_2, G, where G_5 and G_2 are to each other as 5 : 2 and G is any connecting gear—be used, the same curve is obtained. In Fig. 1, G_5 is to replace G_1; G_3 remains unchanged; and G is between G_5 and G_2. This combination gives the equation

$$\alpha\mu t^5 - \alpha t^4 y + \alpha t^3 - t^2 + xt - \mu = 0. \tag{3}$$

By means of this double generation, every type of the curve (1), arising from varying μ and α, can be drawn. For example, from (1) it is obvious that, when $\mu = 0$, we have the equation of the hypocycloid of five cusps; and from (3), when $\mu = 0$, we obtain the cardioid, a curve of one cusp. These two are the limiting forms of the curve and we shall see that the curve varies from one to five cusps.

It seems well to give a name to the curve (1). For several reasons the name Pentastroid seems appropriate, and so it will be used uniformly in what follows to designate the curve (1). In the particular case where the curve is regular, that is, when $\mu = 0$, I shall retain the ordinary usage and refer to it as a five-cusped hypocycloid.

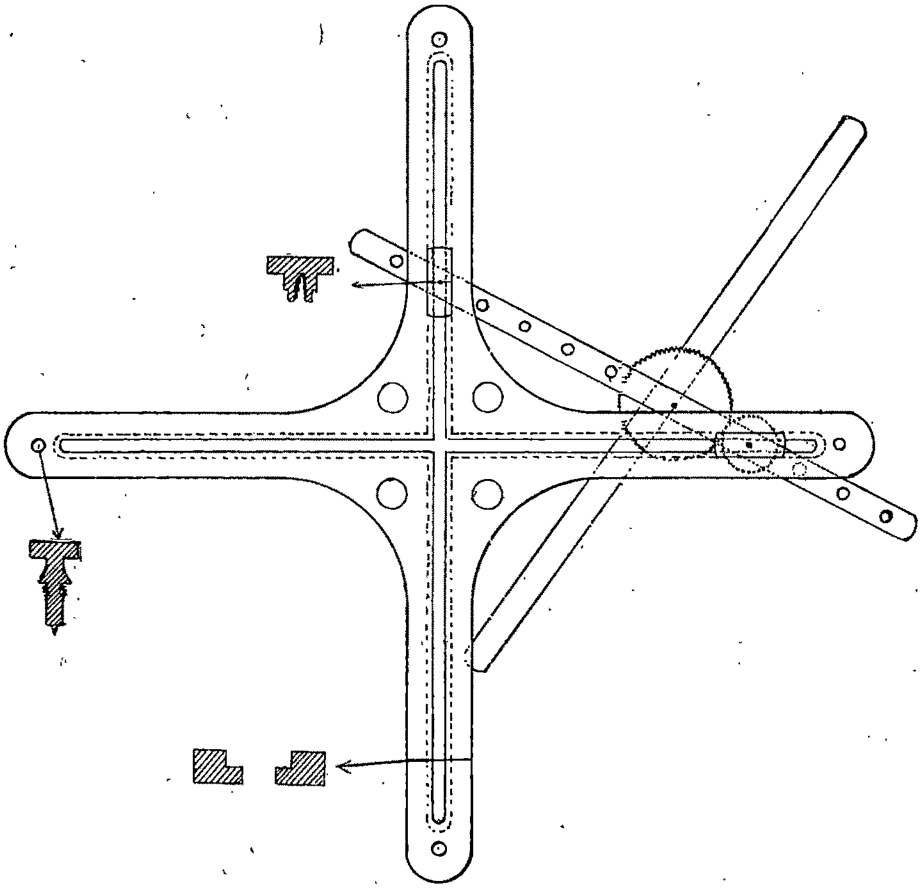

FIG. 2.

III. *The Equation.*

1. Let us consider the equation

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha yt - \alpha = 0, \tag{1}$$

which as t varies envelops a curve of *class five*, for from every point there are five tangents to the curve. If (1) be divided by t and then differentiated with respect to t, there results the equation,

$$3x = 4t + 2\mu/t - \alpha\mu/t^2 + \alpha/t^4, \qquad (4)$$

the map equation of (1), from which it is seen that the curve is of the *eighth degree.*

2. If, in equation (1), the point x be allowed to move off to infinity, the equation reduces to the form

$$t^3 = \alpha y/x. \qquad (5)$$

But y/x is a turn, hence $\alpha y/x$ is a turn; therefore, for x at infinity in any direction, there are three tangents to the curve.

Let the roots of (5) be t_1, ωt_1, $\omega^2 t_1$; then, on substituting these values of t in equation (4) and adding, we obtain

$$\Sigma x_i = 0,$$

where x_i are the points of tangency of these parallel tangents. Therefore,

To a pentastroid there are, in every direction, three parallel tangents, the centroid of whose points of tangency is constant.

In a similar manner, it is proved that

Every curve whose equation has the form

$$t^n + xt^{n-1} + a_1 t^{n-2} + a_2 t^{n-3} + \ldots . + b_2 t^3 + b_1 t^2 + yt + 1 = 0,$$

has $n - 2$ *parallel tangents in any direction, and the centroid of their points of tangency is constant.*

This fixed point is defined as the center of the curve. For the curve with equation as given the center is the origin.

3. We have seen that the map equation of the basic ellipse is

$$x = t + \mu/t. \qquad (2)$$

The clinant* of the tangent to this ellipse is

$$dx/dy = \frac{t^2 - \mu}{\mu t^2 - 1}.$$

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*F. Franklin: Some Applications of Circular Coordinates, AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), p. 162.

- The clinant of the tangent to the curve (4) at the point which has the same parameter is

$$dx/dy = \frac{\alpha}{t^3}.$$

These clinants are equal, that is the tangents are parallel, when

$$t^5 - \mu t^3 - \alpha\mu t^2 + \alpha = 0, \qquad (6)$$

but this is the condition that the two points of tangency shall coincide. Therefore,

The pentastroid touches its basis ellipse in five points, real or imaginary. (Fig. 3.)

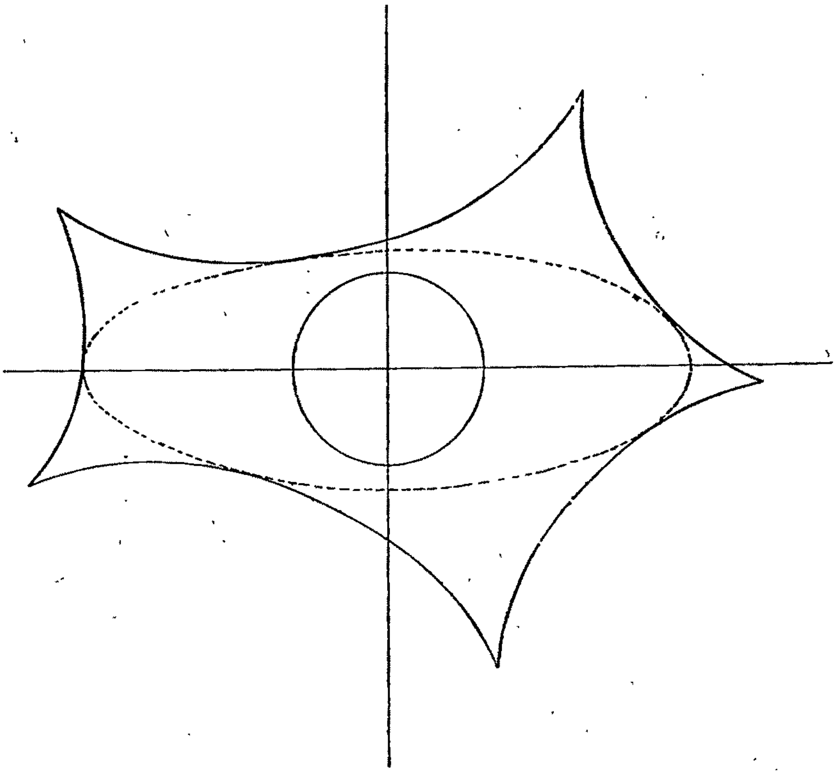

FIG. 3. $\mu = .5$ (nearly).

In the particular case where $\mu = 1$, equation (6) can be solved, and the roots are ± 1, t_1, ωt_1, $\omega^2 t_1$, where t_1 is one of the cube roots of α. Putting these values in the map equation (4), we obtain as the points of tangency

$$x_1 = 2, \qquad x_2 = -2,$$
$$x_3 = t_1 + 1/t_1, \qquad x_4 = \omega t_1 + \omega^2/t_1,$$
$$x = \omega^2 t_1 + \omega/t_1.$$

The ellipse in this case is the segment of the line joining $+2$ and -2. So the pentastroid passes through the points ± 2 for every value of α. The three other points of tangency are on the axis of reals. Hence we see that

When $\mu = 1$, the pentastroid passes through the points ± 2, and the axis of reals is a triple tangent. (Fig. 4.)

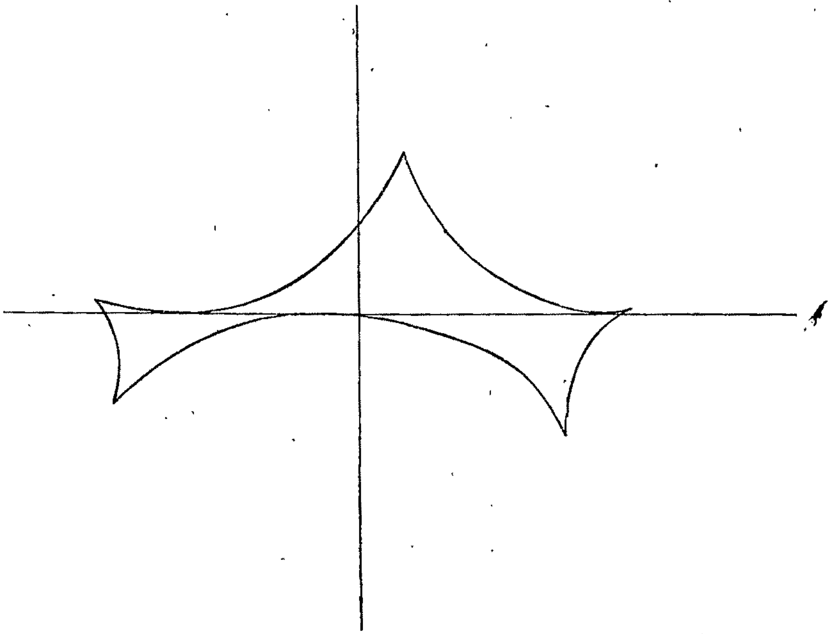

FIG. 4. $\mu = 1$.

4. *Orthoptic Curve.* To the tangent

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha y t - \alpha = 0,$$

the tangents obtained by giving t the values $-t$, $-\omega t$, $-\omega^2 t$, will be perpendicular. Substituting the first of these values, we obtain

$$t^5 + xt^4 + \mu t^3 + \alpha\mu t^2 + \alpha y t + \alpha = 0.$$

If this equation be subtracted from the one above, there results

$$x = -\alpha(\mu/t^2 + 1/t^4), \tag{7}$$

which is the map equation of a limaçon. By means of the same method as that used in connection with the basic ellipse, it is easily shown that this limaçon touches the pentastroid in five points whose parameters are given by the equation

$$2t^5 + \mu t^3 + \alpha\mu t^2 + 2\alpha = 0. \tag{8}$$

When $\mu = 2$, this equation can be solved and its roots are $\pm i$, t_1, ωt_1, $\omega^2 t_1$, where t_1 is one of the cube roots of $-\alpha$. Substituting these values in (7), we obtain as the five points of tangency: α (which is counted twice and is therefore a node), $\mu t_1 + 1/t_1$, $\mu\omega t_1 + \omega^2/t_1$, $\mu\omega^2 t_1 + \omega/t_1$. From equation (4) it is seen that $x = \alpha$ is also a node of the pentastroid.

On substituting $-\omega t$ and $-\omega^2 t$ for t in equation (1) and finding the intersections of the resulting equations with (1), we have

$$2x = (1 - \omega)t + \mu(1 - \omega^2)/t + \alpha\mu\omega^2/t^2 + \alpha\omega/t^4 \qquad (9)$$

and

$$2x = (1 - \omega^2)t + \mu(1 - \omega)/t + \alpha\mu\omega/t^2 + \alpha\omega^2/t^4,$$

respectively. These two forms are easily shown to represent the same curve, for, when t in the second becomes $-\omega t$, the first results.

This curve also touches the pentastroid in five points. The parameters of these points of tangency are given by the equation

$$(5 + 3\omega)t^5 + \mu(1 + 3\omega^2)t^3 + \alpha\mu(3\omega + 1)t^2 + \alpha(5 + 3\omega^2) = 0. \qquad (10)$$

The orthoptic locus of the pentastroid is composed of a limaçon and the curve (9).

5. *Singularities at Infinity.*[*] By comparing equation (1) with the equation $ux + vy = 1$, where u and v are $1/a$ and $1/b$ respectively (a being the reflexion of the origin in the line), the equation of the curve in line coordinates is derived as follows: Equating coefficients, we obtain

$$u = -t^4,$$
$$v = \alpha t,$$
$$w = t^5 + \mu t^3 - \alpha\mu t^2 - \alpha,$$

where w is introduced for the sake of homogeneity. If t be eliminated from these three equations in such a way as to form a homogeneous equation in u, v, and w, the resulting equation is the one required. It is

$$\alpha uvw^3 - [\alpha^2 u^2 (u + \mu v)^3 + v^2 (\mu u + v)^3 + 3uvw (\mu u + v)(u + \mu v)] = 0. \qquad (11)$$

This equation can be transformed to Boothian[†] coordinates by the substitution

$$2u = \xi + i\eta, \qquad 2v = \xi - i\eta.$$

Since the coordinates of the line at infinity (0, 0, 1) satisfy equation (11), the curve is tangent to the line at infinity. The points I and J, whose equations are $u = 0$ and $v = 0$ respectively, are such singular points of the curve that the tangent at each has contact of the third order. This tangent is the line at infinity, therefore the line at infinity is a double tangent.

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[*] F. Franklin: *loc. cit.*, pp. 161–190.

[†] Bassett: *Elementary Treatise on Cubic and Quartic Curves*, p. 30.

The pentastroid is a curve which has the line at infinity for a double tangent whose points of tangency are the points I and J, at each of which the contact is of the third order.

From this it follows that *all the foci of the curve are at infinity.*

6. *Cusps.* The curve will have cusps when $dx/dy = 0$, provided the roots of the resulting equation are turns. Thus we find the parameters of cusps are the roots of the equation

$$2t^5 - \mu t^3 + \alpha\mu t^2 - 2\alpha = 0. \tag{12}$$

This equation may have five turns for roots, hence we say in general that *the pentastroid has five cusps, real, coincident, or imaginary.* (Fig. 3.)

On combining equation (1) with equation (12), we obtain

$$3t^5 - xt^4 + \alpha yt - 3\alpha = 0$$

and

$$2xt^3 - 3\mu t^2 + 3\alpha\mu t - 2\alpha y = 0,$$

of which the first is the equation of a regular pentastroid, *i. e.*, a five-cusped hypocycloid, and the second is the equation of a cardioid. Both are concentric with (1). Whence we have the theorem,

The five cusp-tangents of a pentastroid touch a concentric five-cusped hypocycloid, and also a concentric cardioid.

If x and y be written for μ in (12), thus,

$$2t^5 - xt^3 + \alpha yt^2 - 2\alpha = 0, \tag{13}$$

it is obvious that for those values of x (*i. e.* μ) on the axis of reals from which five tangents can be drawn to the regular pentadeltoid (13), equation (1) has five real cusps; that for those values for which (13) has three tangents only, equation (1) has only three real cusps; and that for those values for which (13) has only one tangent, equation (1) has only one real cusp.

However, in the special case where $\alpha = 1$, more definite limits can be stated for μ. Equation (12) now becomes

$$2(t^5 - 1) - \mu t^2(t - 1) = 0,$$

from which it is seen that $t = 1$ gives a cusp for all values of μ. But since the curve is symmetrical with respect to the axis of reals—when $\alpha = 1$—, if t is a root then $1/t$ is also a root. Suppose then that t_1 and t_2 are roots, then $1/t_1$ and $1/t_2$ are roots also. These relations among the roots make it possible to solve the equation. From the symmetric functions we derive

$$2a = -1 \pm \sqrt{5 + 2\mu}, \tag{14}$$

where $a \equiv t_1 + 1/t_1$.

Evidently a is real and less than, or equal to, 2 in absolute value when t is a turn; hence, in order that a may be real we must have

$$\mu \geqq -5/2.$$

The value of a will be less than, or equal to, 2 when

$$1) \qquad -1 + \sqrt{5 + 2\mu} \leqq 4,$$

that is, when $\mu \leqq 10$; or

$$2) \qquad |-1 - \sqrt{5 + 2\mu}\,| \leqq 4,$$

that is, when $\mu \leqq 2$. From which we conclude

For all values of μ such that $-5/2 \leqq \mu \leqq 2$, the pentastroid for $\alpha = 1$ has five cusps; for all values of μ such that $2 < \mu \leqq 10$, there will be only three real cusps; and for all other values of μ there will be only one real cusp.

The special cases $\mu = -5/2$, 2, 10, when $\alpha = 1$, are interesting. Let us consider first $\mu = -5/2$. Substituting this value of μ in equation (14), and solving for t, we obtain

$$4t = -1 \pm i\sqrt{15},$$

each of which is repeated; that is to say, there are two pairs of coincident cusps. (Fig. 5.)

If $\mu = 10$ when $\alpha = 1$, equation (12) reduces to the form

$$(t - 1)^3\,(t^2 + 3t + 1) = 0,$$

of which $t = 1$ is a repeated root. Hence three of the cusps coincide. The other roots are

$$2t = -3 \pm \sqrt{5}.$$

These are not turns and hence do not give real cusps. In Fig. 6 is seen how the cusps tend to disappear as μ increases.

If $\mu = 2$, equation (11) is solvable for all values of α. It takes the form

$$(t^3 + \alpha)\,(t^2 - 1) = 0,$$

of which the roots are ± 1, t_1, ωt_1, $\omega^2 t_1$, where t_1 is one of the cube roots of $-\alpha$.

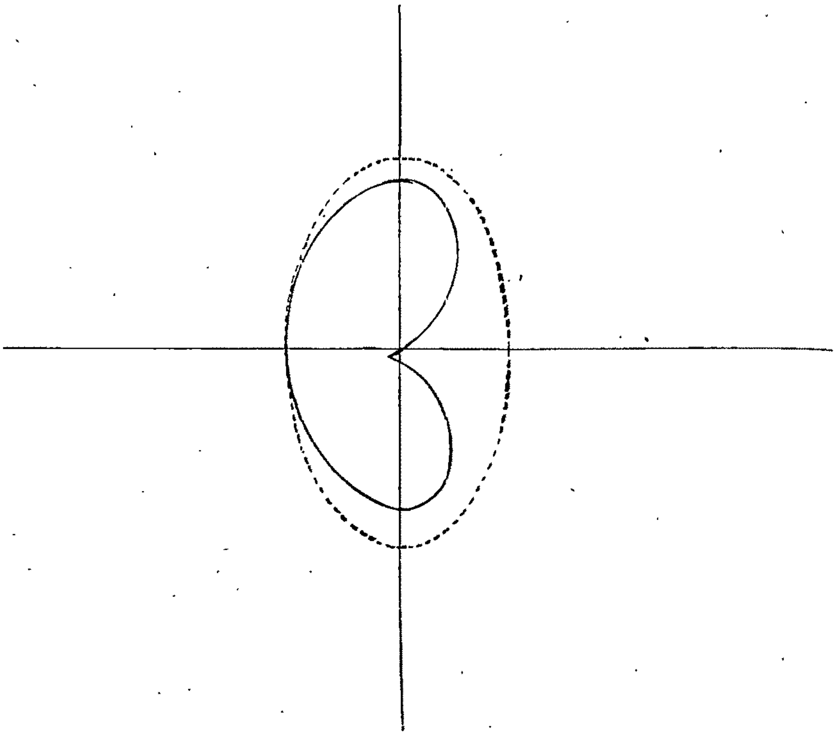

FIG. 5. $\mu = -5/2$.

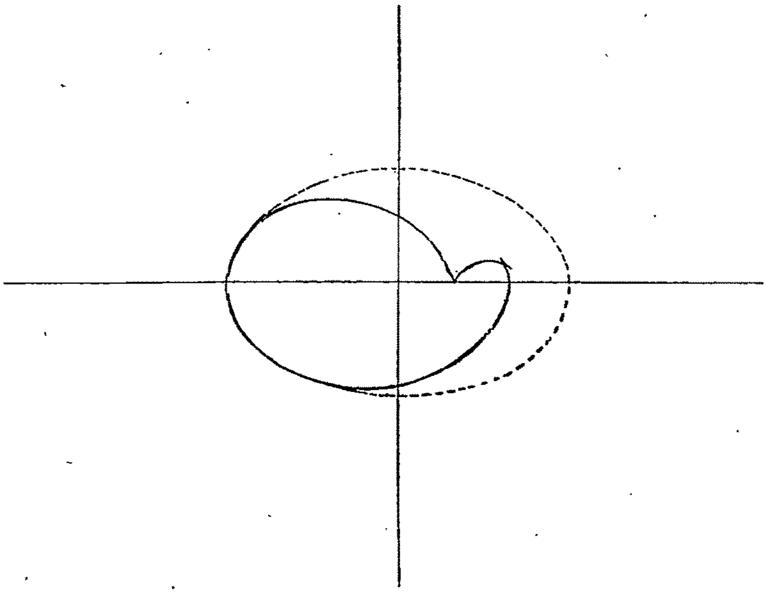

FIG. 6. $\mu = 5$.

Substituting these values of t in equation (4), we find that the five cusps are as follows:

$$
\begin{aligned}
x_1 &= 2\ 2/3 - 1/3\ \alpha, \\
x_2 &= -2\ 2/3 - 1/3\ \alpha, \\
x_3 &= 2t_1 + 1/t_1, \\
x_4 &= 2\omega t_1 + \omega^2/t_1, \\
x_5 &= 2\omega^2 t_1 + \omega/t_1.
\end{aligned}
$$

Now as α varies, the first cusp traces out a circle with radius 1/3 and with center at 2 2/3; the second traces out a circle with the same radius and with center at $-2\ 2/3$; the third, fourth, and fifth trace out the same curve—the basic ellipse of the pentastroid. (Figs. 7, 8, 9.)

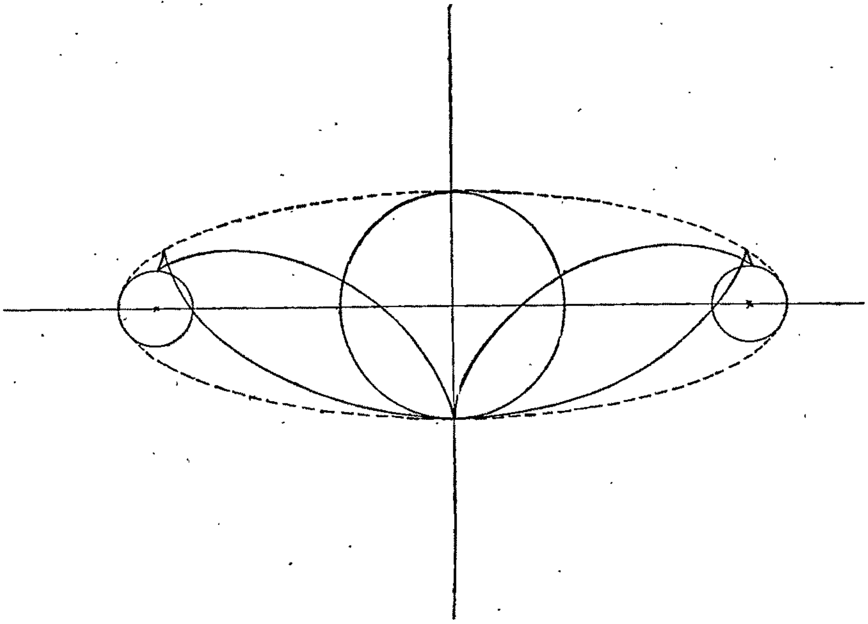

FIG. 7. $\mu = 2.$ $\alpha = i.$

The cusp-tangents at these cusps are:

$$
\begin{aligned}
x - \alpha y &= 3 - 3\alpha, \\
x + \alpha y &= -3 - 3\alpha, \\
x + \ \ y &= 3t_1 + 3/t_1, \\
x + \ \ y &= 3\omega t_1 + 3\omega^2/t_1: \\
x + \ \ y &= 3\omega t_1 + 3\omega^2/t_1:
\end{aligned}
$$

of which the first two are perpendicular lines, passing through the fixed points 3 and -3, respectively; and the last three are parallel lines perpendicular to the axis of reals.

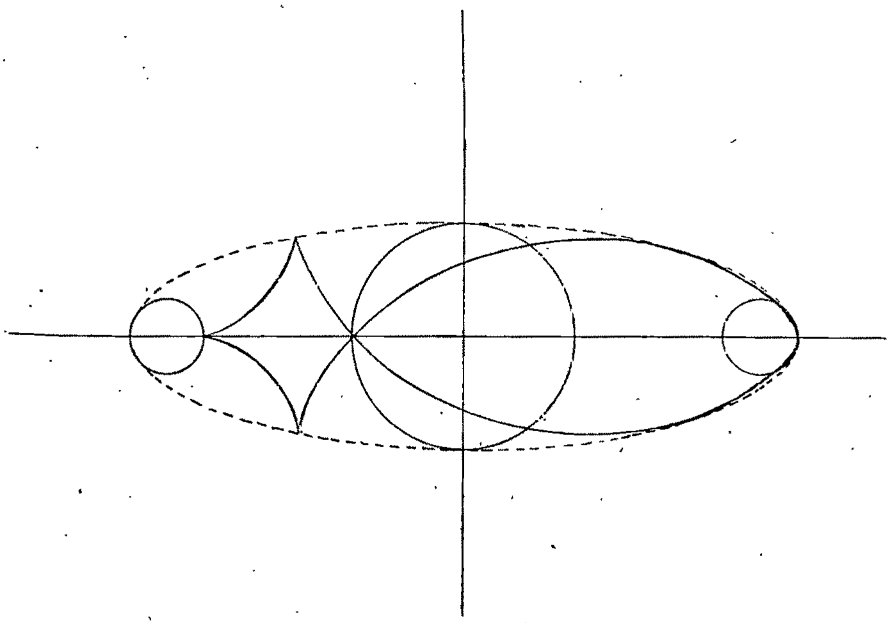

FIG. 8. $\mu = 2.$
$a = -1.$

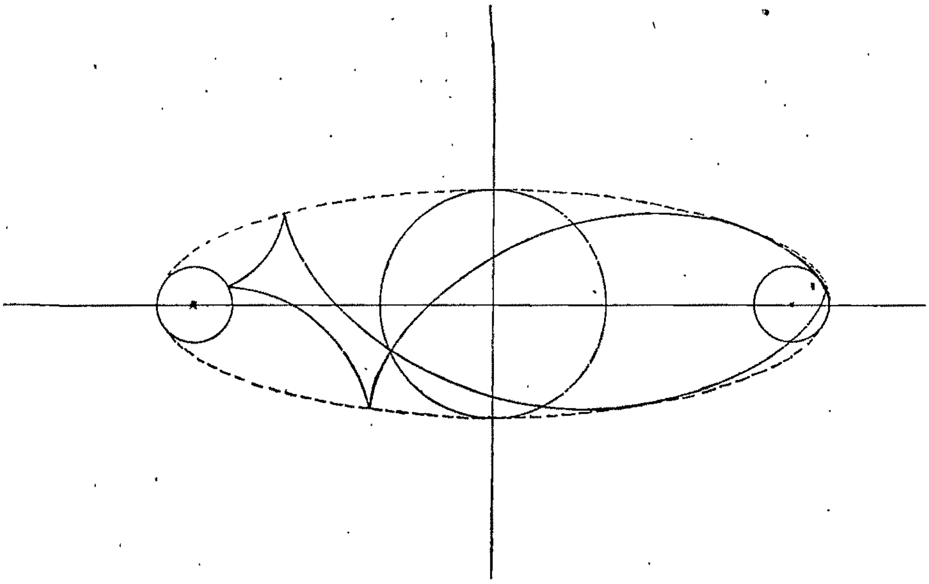

FIG. 9. $\mu = 2.$
$a = e^{i\pi/6}$ [Est.].

7. *Miscellaneous Properties.* In § 4 it was seen that when $\mu = 2$, then

$$x = \alpha$$

is a node of the pentastroid. As α varies, this node runs about the unit circle. The tangents at this node are

$$x - i\alpha y = \alpha + i,$$
$$x + i\alpha y = \alpha + i,$$

two lines which are perpendicular; hence, we can say that the curve cuts itself orthogonally at this node. (Figs. 5, 6, 7.)

The equation of the line normal to the tangent

$$t^5 - xt^4 + \mu t^3 - \alpha\mu t^2 + \alpha yt - \alpha = 0$$

at its point of tangency is

$$5t^5 - xt^4 + \mu t^3 + \alpha\mu t^2 - \alpha yt - 5\alpha = 0,$$

a line which envelops another pentastroid concentric with the first. Hence, we have the theorem

The evolute of the pentastroid is a concentric pentastroid.

Since all the foci of the pentastroid are at infinity, it follows immediately from Laguerre's theorem* that *the sum of the reciprocals of the tangents to the curve from any point is zero.*

WESLEYAN UNIVERSITY, *May* 28, 1907.

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* *Oeuvres de Laguerre*, Vol. II, p. 35.

A Table of the Values of m *Corresponding to Given Values of* $\phi(m)$.*

BY R. D. CARMICHAEL.†

$\phi(m)$	m		
1	1	2	
2	3	4	6
4	5	8	10
	12		
6	7	9	14
	18		
8	15	16	20
	24	30	
10	11	22	
12	13	21	26
	28	36	42
16	17	32	34
	40	48	60
18	19	27	38
	54		
20	25	33	44
	50	66	
22	23	46	
24	35	39	45
	52	56	70
	72	78	84
	90		
28	29	58	
30	31	62	
32	51	64	68
	80	96	102
	120		

$\phi(m)$	m		
36	37	57	63
	74	76	108
	114	126	
40	41	55	75
	82	88	100
	110	132	150
42	43	49	86
	98		
44	69	92	138
46	47	94	
48	65	104	105
	112	130	140
	144	156	168
	180	210	
52	53	106	
54	81	162	
56	87	116	174
58	59	118	
60	61	77	93
	99	122	124
	154	186	198
64	85	128	136
	160	170	192
	204	240	
66	67	134	
70	71	142	

$\phi(m)$	m		
72	73	91	95
	111	117	135
	146	148	152
	182	190	216
	222	228	234
	252	270	
78	79	158	
80	123	164	165
	176	200	220
	246	264	300
	330		
82	83	166	
84	129	147	172
	196	258	294
88	89	115	178
	184	230	276
92	141	188	282
96	97	119	153
	194	195	208
	224	238	260
	280	288	306
	312	336	360
	390	420	
100	101	125	202
	250		
102	103	206	

---

* The object of this table is to give all values of m corresponding to every possible value of Euler's ϕ-function of m up to $\phi(m) = 1000$. The table has been double checked up to $\phi(m) = 500$. The greater portion of the succeeding part of the table may be derived from this part in a simple way. It is therefore believed that but very few errors will be found in the table.

† Read before the American Mathematical Society (Chicago), March 30, 1907.

$\phi(m)$	m		
104	159	212	318
106	107	214	
108	109	133	171
	189	218	266
	324	342	378
110	121	242	
112	113	145	226
	232	290	348
116	177	236	354
120	143	155	175
	183	225	231
	244	248	286
	308	310	350
	366	372	396
	450	462	
126	127	254	
128	255	256	272
	320	340	384
	408	480	510
130	131	262	
132	161	201	207
	268	322	402
	414		
136	137	274	
138	139	278	
140	213	284	426
144	185	219	273
	285	292	296
	304	315	364
	370	380	432
	438	444	456
	468	504	540
	546	570	630
148	149	298	
150	151	302	
156	157	169	237
	314	316	338
	474		

$\phi(m)$	m		
160	187	205	328
	352	374	400
	410	440	492
	528	600	660
162	163	243	326
	486		
164	249	332	498
166	167	334	
168	203	215	245
	261	344	392
	406	430	490
	516	522	588
172	173	346	
176	267	345	356
	368	460	534
	552	690	
178	179	358	
180	181	209	217
	279	297	362
	418	434	558
	594		
184	235	376	470
	564		
190	191	382	
192	193	221	291
	357	386	388
	416	442	448
	476	520	560
	576	582	612
	624	672	714
	720	780	840
196	197	394	
198	199	398	
200	275	303	375
	404	500	550
	606	750	
204	309	412	618
208	265	424	530
	636		
210	211	422	
212	321	428	642

$\phi(m)$	m		
216	247	259	327
	333	351	399
	405	436	494
	518	532	648
	654	666	684
	702	756	798
	810		
220	253	363	484
	506	726	
222	223	446	
224	339	435	452
	464	580	678
	696	870	
226	227	454	
228	229	458	
232	233	295	466
	472	590	708
238	239	478	
240	241	287	305
	325	369	385
	429	465	482
	488	495	496
	525	572	574
	610	616	620
	650	700	732
	738	744	770
	792	858	900
	924	930	990
	1050		
250	251	502	
252	301	381	387
	441	508	602
	762	774	882
256	257	512	514
	544	640	680
	768	816	960
	1020		
260	393	524	786
262	263	526	

$\phi(m)$	m		
264	299	335	483
	536	598	644
	670	804	828
	966		
268	269	538	
270	271	542	
272	289	411	548
	578	822	
276	277	329	417
	423	554	556
	658	834	846
280	281	319	355
	562	568	638
	710	852	
282	283	566	
288	323	365	455
	459	555	584
	585	592	608
	646	728	730
	740	760	864
	876	888	910
	912	918	936
	1008	1080	1092
	1110	1140	1170
	1260		
292	293	586	
294	343	686	
296	447	596	894
300	341	453	604
	682	906	
306	307	614	
310	311	622	
312	313	371	395
	471	477	507
	626	628	632
	676	742	790
	942	948	954
	1014		
316	317	634	

$\phi(m)$	m		
320	425	561	615
	656	704	743
	800	820	850
	880	984	1056
	1122	1200	1230
	1320		
324	489	513	567
	652	972	973
	1026	1134	
328	415	664	830
	996		
330	331	662	
332	501	668	1002
336	337	377	609
	645	674	683
	735	754	784
	812	860	980
	1032	1044	1176
	1218	1290	1470
342	361	722	
344	519	692	1038
346	347	694	
348	349	413	531
	698	826	1062
352	353	391	445
	706	712	736
	782	890	920
	1068	1104	1380
356	537	716	1074
358	359	718	
360	403	407	427
	475	543	549
	627	651	675
	693	724	806
	814	836	854
	868	950	1086
	1098	1116	1188
	1254	1302	1350
	1386		

$\phi(m)$	m		
366	367	734	
368	705	752	940
	1128	1410	
372	373	746	
378	379	758	
380	573	764	1146
382	383	766	
384	485	579	595
	663	765	772
	776	832	884
	896	952	970
	1040	1120	1152
	1158	1164	1190
	1224	1248	1326
	1344	1428	1440
	1530	1560	1680
388	389	778	
392	591	788	1182
396	397	437	469
	597	603	621
	794	796	874
	938	1194	1206
	1242		
400	401	451	505
	802	808	825
	902	1000	1010
	1100	1212	1500
	1650		
408	409	515	818
	824	1030	1236
416	795	848	1060
	1272	1590	
418	419	838	
420	421	473	497
	539	633	639
	842	844	946
	994	1078	1266
	1278		

$\phi(m)$	m		
424	535	856	1070
	1284		
430	431	862	
432	433	481	511
	545	657	665
	741	777	819
	855	866	872
	945	962	988
	1022	1036	1064
	1090	1296	1308
	1314	1330	1332
	1368	1404	1482
	1512	1554	1596
	1620	1638	1710
	1890		
438	439	878	
440	575	605	759
	968	1012	1150
	1210	1452	1518
442	443	886	
444	669	892	1338
448	449	493	565
	898	904	928
	986	1130	1160
	1356	1392	1740
452	681	908	1362
456	457	687	914
	916	1374	
460	461	517	922
	1034		
462	463	926	
464	699	885	932
	944	1180	1398
	1416	1770	
466	467	934	
468	553	711	1106
	1422		

$\phi(m)$	m		
476	717	956	1434
478	479	958	
480	527	533	715
	723	861	915
	964	975	976
	992	1054	1066
	1144	1148	1155
	1220	1232	1240
	1300	1400	1430
	1446	1464	1476
	1488	1540	1584
	1716	1722	1800
	1830	1848	1860
	1950	1980	2100
	2310		
486	487	729	974
	1458		
490	491	982	
492	581	747	1162
	1494		
498	499	998	
500	625	753	1004
	1250	1506	
502	503	1006	
504	551	559	635
	637	783	903
	1016	1102	1118
	1204	1270	1274
	1524	1548	1566
	1764	1806	
506	529	1058	
508	509	1018	
512	771	1024	1028
	1088	1280	1360
	1536	1542	1632
	1920	2040	

$\phi(m)$	m		
520	521	583	655
	1042	1048	1166
	1310	1572	
522	523	1046	
524	789	1052	1578
528	623	801	805
	897	1005	1035
	1072	1196	1246
	1288	1340	1602
	1608	1610	1656
	1794	1932	2010
	2070		
536	807	1076	1614
540	541	589	813
	837	891	1082
	1084	1178	1626
	1674	1782	
544	685	867	1096
	1156	1370	1644
	1734		
546	547	1094	
552	611	695	831
	987	1108	1112
	1222	1316	1390
	1662	1668	1692
	1974		
556	557	1114	
560	725	843	957
	1065	1124	1136
	1276	1420	1450
	1686	1704	1914
	2130		
562	563	1126	
564	849	1132	1698
568	569	1138	
570	571	1142	

$\phi(m)$	m			$\phi(m)$	m			$\phi(m)$	m		
576	577	629	679	630	631	1262		676	677	1354	
	873	969	1071	632	951	1268	1902	682	683	1366	
	1095	1154	1168	636	749	963	1498	684	1083	1444	2166
	1184	1216	1258		1926			688	865	1384	1730
	1292	1358	1365	640	641	697	935		2076		
	1456	1460	1480		1275	1282	1312	690	691	1382	
	1520	1728	1746		1394	1408	1496	692	1041	1388	2082
	1752	1776	1820		1600	1640	1700	696	767	1047	1239
	1824	1836	1872		1760	1870	1968		1396	1534	1652
	1938	2016	2142		2112	2244	2400		2094	2124	2478
	2160	2184	2190		2460	2550	2640	700	701	781	1402
	2220	2280	2340	642	643	1286			1562		
	2520	2730		646	647	1294		704	1059	1173	1335
580	649	1298		648	703	763	815		1412	1424	1472
584	879	1172	1758		981	999	1053		1564	1780	1840
586	587	1174			1197	1215	1304		2118	2136	2208
588	1029	1372	2058		1406	1526	1630		2346	2670	2760
592	593	745	1186		1944	1956	1962	708	709	1418	
	1192	1490	1788		1998	2052	2106	712	895	1432	1790
598	599	1198			2268	2394	2430		2148		
600	601	671	707	652	653	1306		716	1077	1436	2154
	755	775	875	656	1245	1328	1660	718	719	1438	
	909	1023	1125		1992	2490		720	779	793	803
	1202	1208	1342	658	659	1318			905	925	1001
	1364	1414	1510	660	661	713	737		1045	1085	1107
	1550	1750	1812		847	993	1089		1209	1221	1281
	1818	2046	2250		1322	1324	1426		1287	1395	1425
606	607	1214			1474	1694	1986		1448	1485	1558
612	613	721	921		2178				1575	1586	1606
	927	1226	1228	664	835	1336	1670		1612	1628	1672
	1442	1842	1854		2004				1708	1736	1810
616	617	667	1234	672	673	731	791		1850	1900	2002
	1334				833	1011	1015		2090	2170	2172
618	619	1238			1017	1131	1305		2196	2214	2232
620	933	1244	1866		1346	1348	1376		2376	2418	2442
624	689	785	845		1462	1508	1568		2508	2562	2574
	939	1113	1185		1582	1624	1666		2604	2700	2772
	1252	1256	1264		1720	1960	2022		2790	2850	2970
	1352	1378	1484		2030	2034	2064		3150		
	1570	1580	1690		2088	2262	2352	726	727	1454	
	1878	1884	1896		2436	2580	2610	732	733	1101	1466
	1908	2028	2226		2940				1468	2202	
	2370										

$\phi(m)$	m		
736	799	1504	1598
	1880	2256	2820
738	739	1478	
742	743	1486	
744	1119	1492	2238
750	751	1502	
756	757	817	889
	931	1137	1143
	1161	1323	1514
	1516	1634	1778
	1862	2274	2286
	2322	2646	
760	761	955	1522
	1528	1910	2292
764	1149	1532	2298
768	769	965	1105
	1455	1538	1544
	1552	1664	1768
	1792	1904	1930
	1940	2080	2210
	2240	2304	2316
	2328	2380	2448
	2496	2652	2688
	2856	2880	2910
	3060	3120	3360
772	773	1546	
776	1167	1556	2334
780	869	917	1179
	1738	1834	2358
784	985	1576	1970
	2364		
786	787	1574	
792	851	871	995
	1191	1311	1407
	1449	1588	1592
	1702	1742	1748
	1876	1990	2382
	2384	2412	2484
	2622	2814	2898
796	797	1594	

$\phi(m)$	m		
800	1025	1203	1353
	1515	1604	1616
	1804	2000	2020
	2050	2200	2406
	2424	2706	3000
	3030	3300	
808	809	1618	
810	811	1622	
812	841	1682	
816	959	1227	1233
	1545	1636	1648
	1918	2060	2454
	2466	2472	3090
820	821	913	1642
	1826		
822	823	1646	
826	827	1654	
828	829	893	973
	1251	1269	1658
	1786	1946	2502
	2538		
832	901	1696	1802
	2120	2544	3180
836	1257	1676	2514
838	839	1678	
840	899	923	1055
	1075	1225	1263
	1419	1491	1617
	1684	1688	1798
	1846	1892	1988
	2110	2150	2156
	2450	2526	2532
	2556	2838	2982
	3234		
848	1605	1712	2140
	2568	3210	
852	853	1706	
856	857	1714	
858	859	1718	
860	1293	1724	2586
862	863	1726	

$\phi(m)$	m		
864	949	1235	1295
	1299	1377	1443
	1533	1635	1665
	1732	1744	1755
	1898	1924	1976
	1995	2044	2072
	2128	2180	2470
	2590	2592	2598
	2616	2628	2660
	2664	2736	2754
	2808	2886	2964
	3024	3066	3108
	3192	3240	3270
	3276	3330	3420
	3510	3780	3990
876	877	1317	1754
	1756	2634	
880	881	943	979
	1043	1265	1725
	1762	1815	1886
	1936	1958	2024
	2086	2300	2420
	2530	2904	3036
	3450	3630	
882	883	1766	
884	1329	1772	2658
886	887	1774	
888	1115	1341	1784
	2230	2676	2682
896	1347	1479	1695
	1796	1808	1856
	1972	2260	2320
	2694	2712	2784
	2958	3390	3480
900	1057	1359	2114
	2718		
904	1135	1816	2270
	2724		
906	907	1814	
910	911	1822	

$\phi(m)$	m		
912	1145	1371	1828
	1832	2290	2742
	2748		
918	919	1838	
920	1175	1383	1551
	1844	2068	2350
	2766	3102	
924	989	1127	1389
	1852	1978	2254
	2778		
928	929	1003	1165
	1858	1864	1888
	2006	2330	2360
	2796	2832	3540
930	961	1922	
932	1401	1868	2802
936	937	1007	1027
	1099	1183	1413
	1431	1521	1659
	1874	2014	2054
	2198	2212	2366
	2826	2844	2862
	3042	3318	
940	941	1882	
946	947	1894	
952	953	1195	1906
	1912	2390	2868
956	1437	1916	2874
960	1037	1067	1205
	1435	1531	1599
	1683	1845	1928
	1952	1934	2074
	2108	2132	2134
	2145	2288	2296
	2410	2440	2464
	2480	2600	2800
	2860	2870	2892
	2928	2952	2976
	3080	3162	3168
	3198	3366	3432
	3444	3600	3660
	3690	3696	3720
	3900	3960	4200
	4290	4620	
966	967	1934	
970	971	1942	
972	1141	1461	1539
	1701	1948	2282
	2916	2922	3078
	3402		
976	977	1954	
980	1473	1964	2946
982	983	1966	
984	1079	1743	2158
	2324	2988	3486
990	991	1982	
996	997	1169	1497
	1503	1994	1996
	2338	2994	3006
1000	1111	1255	1375
	1875	2008	2222
	2500	2510	2750
	3012	3750	

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